SI 2014 Q1

(i) $\boxed{B1}$ for $3 = 2^2 - 1^2$
$5 = 3^2 - 2^2$
$8 = 3^2 - 1^2$
$12 = 4^2 - 2^2$
$16 = 5^2 - 3^2$ 1

(ii) $\boxed{B1}$ for $2n+1 \equiv (n+1)^2 - n^2$ OR $2n-1 \equiv n^2 - (n-1)^2$ 1

(iii) $\boxed{B1}$ for $(k+1)^2 - (k-1)^2 \equiv 4k$ 1

(iv) APPROACH 1 – parity argument (4 cases to consider)

odd^2 – even^2 and even^2 – odd^2 = odd $\boxed{B1}$ $\boxed{B1}$

[ALT. $a^2 - b^2$ is even iff a, b both odd or both even B2]

odd^2 – odd^2 $= (2a+1)^2 - (2b+1)^2$ $\boxed{M1}$ attempted

$= 4a^2 + 4a - 4b^2 - 4b$ $\boxed{A1}$
OR $4(a^2 + a - b^2 - b)$

even^2 – even^2 $= \underbrace{(2a)^2 - (2b)^2}_{\boxed{M1}} = 4a^2 - 4b^2$ $\boxed{A1}$
OR $4(a^2 - b^2)$

∴ Every even difference of 2 squares is of the form $4k$ and not $4k+2$ $\boxed{E1}$ MUST BE STATED; NOT TO BE GIVEN UNLESS THEY HAVE FULLY SHOWN IT SO.

APPROACH 2 – modular arithmetic

$x^2 \pmod 4 \equiv 0, 1$ $\boxed{M2}$ $\boxed{A1}$

$\Rightarrow a^2 - b^2 \pmod 4 \equiv 1, 0, -1 (\equiv 3)$ $\boxed{A1}$ $\boxed{A1}$ $\boxed{A1}$

So $a^2 - b^2 \not\equiv 2 \pmod 4$ $\boxed{E1}$ MUST BE STATED ..... etc. 7

(v) $m^2-n^2=(m+n)(m-n)=pq$ $\boxed{M1}$ Diff. of 2sqs. factsn. ($p>q$ here)

$\Rightarrow p=m+n,\ q=m-n$ $\boxed{A1}$

OR

$pq=m+n,\ 1=m-n$ $\boxed{A1}$

Since p,q (odd) primes, these are the only two factorisations of pq
$\Rightarrow$ exactly two ways $\boxed{E1}$

If $q=2$ we have $2p=m^2-n^2$ where p is odd
so this is of the form $4k+2$ $\boxed{B1}$
and (iv) $\Rightarrow$ this is impossible $\boxed{E1}$

6

(vi) $675=3^3.5^2$ $\boxed{M1}$ Prime factorisation attempt $\boxed{A1}$ correct

$\Rightarrow (3+1)(2+1)=12$ factors $\boxed{M1}$

$\Rightarrow$ 6 factor-pairs $\boxed{A1}$

~~ALT~~ for 2nd M1 A1 Listing factors

675	225	135	75	45	27
1	3	5	9	15	25

PLEASE BE VERY STRICT WITH THESE LAST 4 MARKS.
FOR FINAL A1 THEY MUST INDICATE CLEARLY
THAT THEY KNOW THE REQUIRED ANSWER IS 6

4

NB $675=(m+n)(m-n)=ab \Rightarrow m+n=a,\ m-n=b$
$\Rightarrow m=\frac{1}{2}(a+b),\ n=\frac{1}{2}(a-b)$

and so the six possibilities are

$675=338^2-337^2=114^2-111^2=70^2-65^2$

$=42^2-33^2=30^2-15^2=26^2-1^2$

NOT ASKED-FOR

M0 M0 for "search" methods, even if all 6 found, without proper justification

SI 2014 Q2

(i) Method 1 – differentiation

$$y = -(2-x).\ln(2-x) + (2-x) + c \qquad (x<2)$$

$$\Rightarrow \frac{dy}{dx} = \underbrace{-(2-x).\frac{1}{2-x}.-1}\ + \ln(2-x).1 \ - 1$$

$\boxed{M1}$ Good use of the PRODUCT RULE

$\boxed{M1}$ Good use of the CHAIN RULE $\quad \underbrace{\boxed{A1}}$ unsimplified

$$= 1 + \ln(2-x) - 1$$

$$= \ln(2-x)$$ $\boxed{A1}$ all correct and simplified

Method 2 – integration

$$\int \ln(2-x).1\,dx$$ $\boxed{M1}$ Use of parts $\qquad (x<2)$

$$= x.\ln(2-x) - \int \frac{-1}{2-x}.x\,dx$$ $\boxed{A1}$

$$= x\ln(2-x) - \int \frac{2-x-2}{2-x}\,dx$$

$$= x\ln(2-x) + \int\left(-1 + \frac{2}{2-x}\right)dx$$ $\boxed{M1}$

$$= x\ln(2-x) - x - 2\ln(2-x) + c_1$$

$$= -(2-x)\ln(2-x) + (2-x) + c \qquad (c = c_1 - 2)$$

$\boxed{A1}$ Correctly found GIVEN ANSWER

4

(iii) Area $= \displaystyle\int_0^{\sqrt3} \ln(2-x)\,dx + \int_0^{\sqrt3}\ln(2+x)\,dx$ $\boxed{M1}$ Splitting up log. term

$$= \Big[-(2-x)\ln(2-x) - x\Big]_0^{\sqrt3} + \Big[(2+x)\ln(2+x) - x\Big]_0^{\sqrt3}$$

$\boxed{A1}$ with/without $+2$ $\qquad$ $\boxed{A1}$ ditto the 2

5

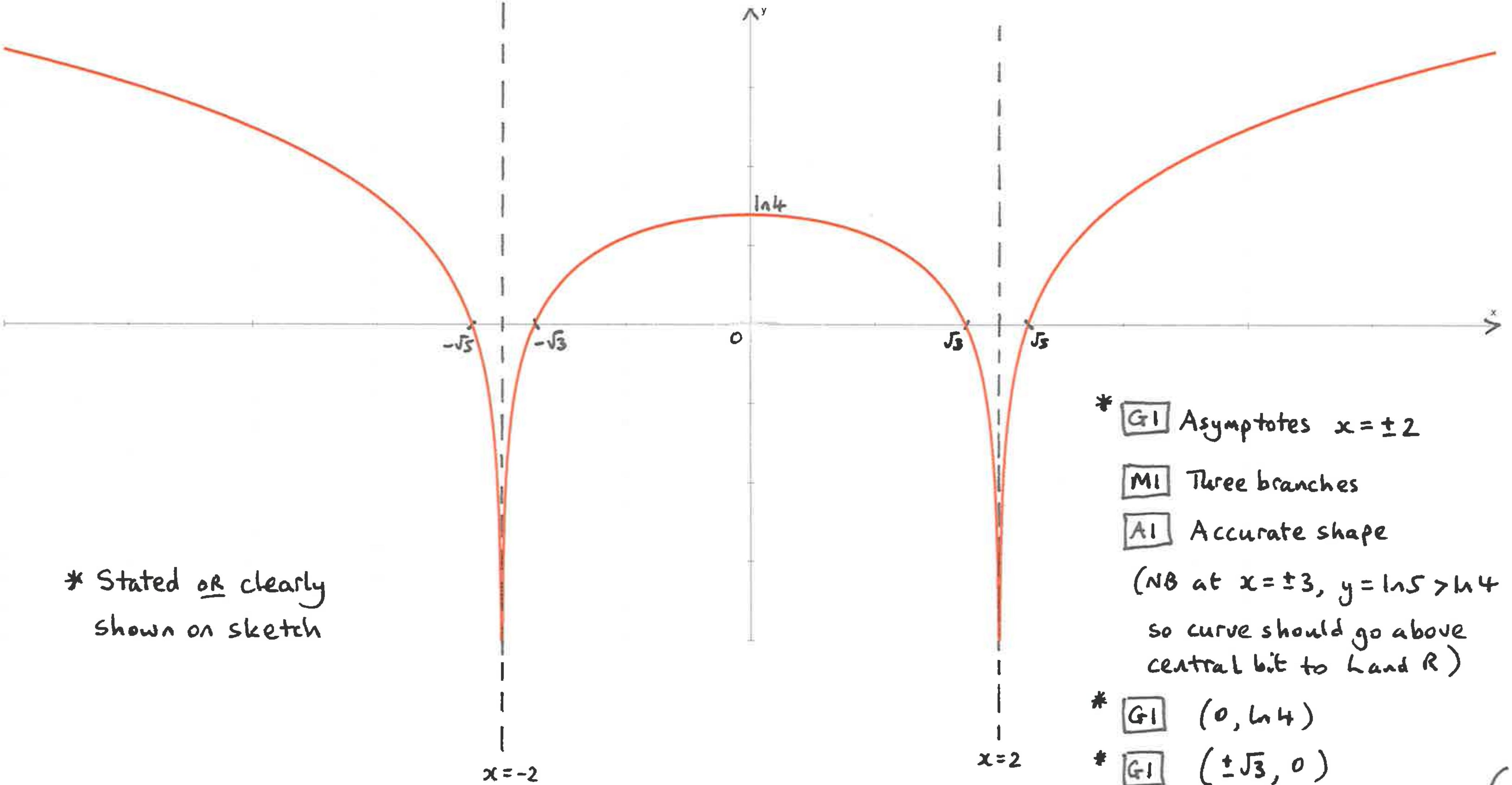
$$= \left(-(2-\sqrt3)\ln(2-\sqrt3) - \sqrt3 + 2\ln2\right) + \left((2+\sqrt3)\ln(2+\sqrt3) - \sqrt3 - 2\ln2\right)$$

$$= (2-\sqrt3)\ln(2+\sqrt3)^{*} - \sqrt3 + (2+\sqrt3)\ln(2+\sqrt3) - \sqrt3$$

$$= 4\ln(2+\sqrt3) - 2\sqrt3$$ $\boxed{A1}$ GIVEN ANSWER correctly obtained

* $\boxed{M1}$ Use of $2-\sqrt3 = (2+\sqrt3)^{-1}$

MUST APPEAR EXPLICITLY



* Stated OR clearly shown on sketch

* $\boxed{G1}$ Asymptotes $x = \pm 2$

$\boxed{M1}$ Three branches

$\boxed{A1}$ Accurate shape

(NB at $x = \pm 3$, $y = \ln 5 > \ln 4$ so curve should go above central bit to L and R)

* $\boxed{G1}$ $(0, \ln 4)$

* $\boxed{G1}$ $(\pm\sqrt{3}, 0)$

* $\boxed{G1}$ $(\pm\sqrt{5}, 0)$

6

(iv) 2nd area $= -\int_{\sqrt{3}}^{2} \{\ln(2-x) + \ln(2+x)\}$

$= \left[(2-x)\ln(2-x) + x\right]_{\sqrt{3}}^{2} - \left[(2+x)\ln(2+x) - x\right]_{\sqrt{3}}^{2}$ $\boxed{M1}$ ± here

$= \left(0^{*} + 2 - (2-\sqrt{3})\ln(2-\sqrt{3}) - \sqrt{3}\right)$

* using $t\ln t \to 0$ as $t \to 0$ $\boxed{A1}$

$- \left(4\ln 4 - 2 - (2+\sqrt{3})\ln(2+\sqrt{3}) + \sqrt{3}\right)$

$\boxed{A1}$

$= 4\ln(2+\sqrt{3}) + 4 - 2\sqrt{3} - 4\ln 4$

Total area is $2\left\{\left(\underbrace{4\ln(2+\sqrt{3}) - 2\sqrt{3}}_{\text{(iii)'s GIVEN ANSWER}}\right) + \left(4\ln(2+\sqrt{3}) + 4 - 2\sqrt{3} - 4\ln 4\right)\right\}$

$\boxed{M1}$

ft above provided signs now correct <u>and</u> $\ln(2-\sqrt{3}) \to -\ln(2+\sqrt{3})$ sorted

Area $= 8\left(2\ln(2+\sqrt{3}) + 1 - \sqrt{3} - \ln 4\right)$ $\boxed{A1}$

5

SI 2014 Q3

(i) LHS $= \left[\dfrac{x^3}{3}\right]_0^b = \dfrac{1}{3}b^3$ $\boxed{B1}$ and RHS $= \left(\left[\dfrac{x^2}{2}\right]_0^b\right)^2 = \dfrac{1}{4}b^4$ $\boxed{B1}$

$b \neq 0 \Rightarrow b = 4/3$ $\boxed{A1}$ CORRECT ANSWER ONLY 3

(ii) $\dfrac{b^3-1}{3} = \left(\dfrac{b^2-1}{2}\right)^2$ M1

$\Rightarrow 4(b-1)(b^2+b+1) = 3(b-1)^2(b+1)^2$ $\boxed{M1}$ creating a quadratic (or cubic)

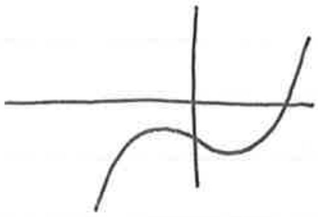
$\Rightarrow 3b^4 - 4b^3 - 6b^2 + 7 = 0$

$\Rightarrow (b-1)(3b^3 - b^2 - 7b - 7) = 0$ 3

$b > a$ so $b \neq 1$ and $3b^3 - b^2 - 7b - 7 = 0$ $\boxed{A1}$ GIVEN ANSWER properly justified

For $y = 3x^3 - x^2 - 7x - 7$, $\dfrac{dy}{dx} = 9x^2 - 2x - 7 = (9x+7)(x-1)$ $\boxed{M1}$

giving TPs at $\left(-\frac{7}{9}, -\frac{864}{243}\right)$ and $(1, -12)$ $\boxed{A1}$ y-coords not required provided they can be shown to be $-$ve



Both TPs below x-axis

$\Rightarrow$ exactly one real (positive) root 3

$\boxed{E1}$ Explanation

ALT: $\dfrac{dy}{dx} = \left(3x - \frac{1}{3}\right)^2 - \dfrac{64}{9} > 0$ for $x > 1$ so $\nexists$ no TPs to right of $x = 1$

$y(2) = -1$ and $y(3) = 44$ B1

"Change-of-Sign" Rule (for a continuous fn.) $\Rightarrow 2 < \text{root} < 3$ $\boxed{E1}$ 2

(iii) In general, we have $\dfrac{b^3-a^3}{3} = \left(\dfrac{b^2-a^2}{2}\right)^2$

$\Rightarrow 4(b-a)(b^2+ab+a^2) = 3(b-a)^2(b+a)^2$ $\boxed{M1}$ MUST factorise

$\boxed{M1}$ p, q substd. ($p = a+b$)

$b > a \Rightarrow q \neq 0 \ (=b-a) \Rightarrow 4(p^2 - ab) = 3qp^2$

NB $p^2 - q^2 = 4ab$ $\boxed{M1}$ $\Rightarrow 4p^2 - (p^2 - q^2) = 3qp^2$ $\boxed{A1}$

$\Rightarrow \underline{\underline{3p^2 + q^2 = 3p^2 q}}$ GIVEN ANSWER legitimately obtained

$\Rightarrow \underline{\underline{p^2 = \dfrac{q^2}{3(q-1)}}}$ $\boxed{B1}$ CAO

5

Since $p^2 > 0$, $q > 1$ from this $\boxed{E1}$

Also, $p^2 - q^2 \geq 0$ $\boxed{E1}$ $\Rightarrow \dfrac{q^2}{3(q-1)} - \dfrac{q^2 . 3(q-1)}{3(q-1)}$ $\boxed{M1}$ $= \dfrac{q^2(4-3q)}{3(q-1)} \geq 0$

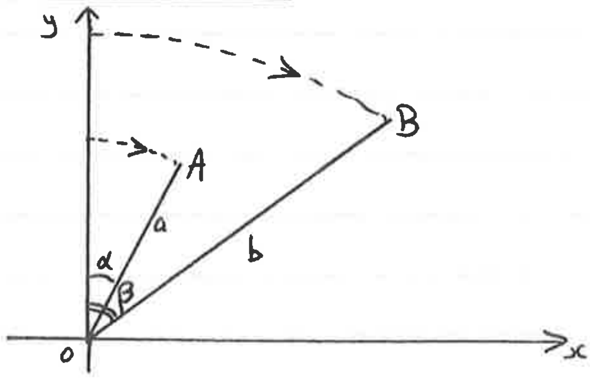
$\Rightarrow q = b - a \leq \frac{4}{3}$ $\boxed{A1}$

4

ALT $p \geq q$ (and both +ve) $\Rightarrow \dfrac{q^2}{3(q-1)} \geq q^2 \Rightarrow \frac{1}{3} \geq q - 1 \Rightarrow \frac{4}{3} \geq q$

(Hence $1 < b - a \leq \frac{4}{3}$, as required)

SI 2014 Q4



Hour-hand A moves at $\frac{2\pi}{12}$ rad/hr $\boxed{M1}$ Set-up

Minute-hand B .. .. 2π .. ..

so at time t hours after noon (w.l.o.g. start with both hands together at noon)

$A = (a\sin\alpha, a\cos\alpha)$ $\quad B = (b\sin\beta, b\cos\beta)$

where $\alpha = kt$ and $\beta = 12kt$ $\boxed{A1}$ $\left\{ k = \frac{\pi}{6} \right\}$

Then $AB^2 = (b\sin 12kt - a\sin kt)^2 + (b\cos 12kt - a\cos kt)^2$ $\boxed{M1}$ $\boxed{A1}$ correct unsimplified

$= b^2 + a^2 - 2ab\{\sin 12kt \sin kt + \cos 12kt \cos kt\}$

use $\cos(P-Q)$ $\boxed{M1}$

$\Rightarrow$ $\underline{\underline{D = (b^2+a^2 - 2ab\cos 11kt)^{1/2}}}$ $\boxed{A1}$ ALT. By the Cosine Rule 6

$\boxed{M1}$ Diff^{g} D $\quad$ $\boxed{M1}$ with good effort at use of CHAIN RULE

Rate of change is $\frac{dD}{dt} = \frac{1}{2}(b^2+a^2-2ab\cos 11kt)^{-1/2} \cdot 2ab\sin 11kt \cdot 11k$

$$\underline{\underline{= 11kab \left\{ \frac{\sin 11kt}{(b^2+a^2-2ab\cos 11kt)^{1/2}} \right\}}}$$

$\boxed{A1}$ Okay unsimplified

(Allow FT if only constant factors are incorrect) 3

This is a maximum when $\frac{d^2D}{dt^2} = 0$ $\boxed{M1}$

NB Just did this already

i.e. (ignoring denominator)

$$(b^2+a^2-2ab\cos 11kt)^{1/2} . 11k\cos 11kt = \sin 11kt . \frac{11kab\sin 11kt}{(b^2+a^2-2ab\cos 11kt)^{1/2}}$$

$\boxed{M1}$ QUOTIENT AND CHAIN RULES APPLIED $\boxed{A1}$

i.e. when $(b^2+a^2-2ab\cos 11kt)\cos 11kt = ab\sin^2 11kt$

$$\Leftarrow \quad (b^2+a^2)C = ab(2C^2+S^2) = ab(1+C^2)$$

$$\Leftarrow \quad abC^2-(b^2+a^2)C+ab = 0 \quad \boxed{M1}$$

$$\Leftarrow \quad (aC-b)(bC-a) = 0 \quad \boxed{M1}$$

Since $b>a$, $\cos 11kt \neq \frac{b}{a}$ so $\cos 11kt = \frac{a}{b}$ $\boxed{E1}$ Explanation of choice for C

* and $D = \left(b^2+a^2-2ab.\frac{a}{b}\right) = (b^2-a^2)^{1/2}$ $\boxed{A1}$ Allow even if E0 above

GIVEN ANSWER

7

When $b=2a$, $\cos\frac{11\pi t}{6} = \frac{1}{2}$ $\boxed{M1}$ $\boxed{A1}$

$$\Rightarrow \quad \frac{11\pi t}{6} = \frac{\pi}{3} \quad (1^{st} \text{ occurrence}) \quad \boxed{M1}$$

$$\Rightarrow \quad t = \frac{2}{11}\text{ hours} = \frac{120}{11} = 10\tfrac{10}{11} \approx 11\text{ mins} \quad \boxed{A1}$$

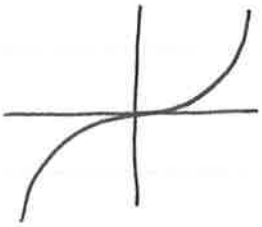
GIVEN ANSWER

4

* Note I am not expecting candidates to show it is a MAX. point rather than a MIN.

SI 2014 Q5

(i) If $a=0$, $f(x)=x^3$

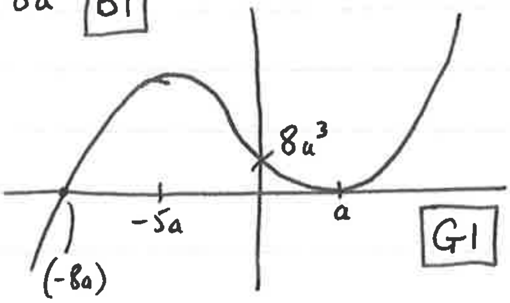


and $f \geq 0 \ \forall x \geq 0$ $\boxed{B1}$ **1**

If $a>0$, $f'(x) = \overset{\boxed{M1}}{3(x+2a)^2 - 27a^2} = \overset{\boxed{M1}}{3\{(x+2a-3a)(x+2a+3a)\}}$

$= 0$ at $(a, 0)$ and $(-5a, 108a^3)$ $\boxed{A1}$

Also, $f(0) = 8a^3$ $\boxed{B1}$



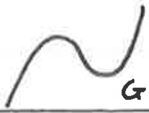
$\boxed{G1}$

giving $f(x) \geq 0 \ \forall x \geq 0$

$\boxed{A1}$ Conclusion MUST be noted **6**

ALT. $f(x) \equiv (x-a)^2(x+8a)$ M1 A1 gives zeroes at $x=-8a$ A1, $x=a$ (twice) A1

and positive cubic shape



G1 $\Rightarrow$ etc. QED. A1

(ii) With $a=y$ $\boxed{M1}$, (i) gives $27xy^2 \leq (x+2y)^3$ $\boxed{A1}$ **4**

Using $x+2y \leq 3$ $\boxed{M1}$ $\Rightarrow$ $xy^2 \leq 1$ $\boxed{A1}$

Equality from (i) occurs when $x=a$ $(=y)$ $\boxed{M1}$ **2**

and $x+2y=3$ gives equality iff $x=y=1$ $\boxed{A1}$

Give B1 for correct result from missing | inconclusive | fudged working

(iii) From (i), $(x+2a)^3 - 27a^2x \geq 0 \quad \forall x \geq 0$

Set $x = p$ and $2a = q+r$ $\boxed{M1}$

$\Rightarrow \quad (p+q+r)^3 - 27\left(\frac{q+r}{2}\right)^2 p \geq 0$ $\boxed{A1}$

* Now $\left(\frac{q+r}{2}\right)^2 \geq qr \Leftrightarrow (q-r)^2 \geq 0$ which is clearly true

$\boxed{M1}$ Attempt to prove this $\boxed{A1}$ Validly done

giving $\underline{\underline{(p+q+r)^3 \geq 27pqr}}$ $\boxed{A1}$

NB Some may simply cite AM ≥ GM to justify this.
Anyone who ignores request to "use part (i)....." and uses the 3-variable AM − GM: $\frac{p+q+r}{3} \geq \sqrt[3]{pqr}$

$\Rightarrow (p+q+r)^3 \geq 27pqr$

give M1 A1 only

5

From (i), the equality case arises when
$x = a$ i.e. $p = \frac{q+r}{2}$ $\boxed{E1}$ Both bits MUST be supplied

and $\left(\frac{q+r}{2}\right)^2 \geq qr$ equality iff $q=r$

giving $p = q = r$ $\boxed{B1}$

2

SI 2014 Q6

(i) $u_{n+1} = 4u_n(1-u_n) \qquad u_0 = u$

$u_0 = \sin^2\theta \Rightarrow u_1 = 4\sin^2\theta(1-\sin^2\theta)$ $\boxed{M1}$

$= 4\sin^2\theta\cos^2\theta = \sin^2 2\theta$ $\boxed{A1}$

$\Rightarrow u_2 = \sin^2 4\theta$ similarly $\boxed{A1}$ 3

Conjecture: $u_n = \sin^2(2^n\theta)$ $\boxed{B1}$

$\Rightarrow u_{n+1} = \sin^2(2.2^n\theta)$ from above $\boxed{M1}$

$= \sin^2(2^{n+1}\theta)$ $\boxed{A1}$ and the result follows by induction 3

NB No need to provide baseline case or explain the inductive method.

(ii) $v_{n+1} = -pv_n^2 + qv_n + r \quad *$ $\quad (p, q, r$ constants; $p \neq 0)$

and $v_0 = v$

$\boxed{M1}$ Substn. of $v_n = \alpha u_n + \beta$ into $*$

$v_{n+1} = \alpha u_{n+1} + \beta = -p(\alpha^2 u_n^2 + 2\alpha\beta u_n + \beta^2) + q(\alpha u_n + \beta) + r$

$= -p\alpha^2 u_n^2 + (q\alpha - 2p\alpha\beta)u_n + (q\beta + r - p\beta^2)$ $\boxed{A1}$

$\Rightarrow u_{n+1} = -p\alpha u_n^2 + (q-2p\beta)u_n + \dfrac{(q-1)\beta + r - p\beta^2}{\alpha}$

$\boxed{M1}$

c.f. $\quad -4u_n^2 + 4u_n$ $\boxed{M1}$

2 correct $\boxed{A1}$ all 3 ✓ + $\boxed{A1}$

$\Rightarrow p\alpha = 4 \qquad q - 2p\beta = 4$ and $(q-1)\beta + r = p\beta^2$

Substg. $\alpha = \dfrac{4}{p}$ and $\beta = \dfrac{q-4}{2p}$ into $\boxed{M1}$

$\Rightarrow \dfrac{(q-1)(q-4)}{2p} + r = \dfrac{p(q-4)^2}{4p^2}$

$\Rightarrow 2(q-1)(q-4) + 4pr = (q-4)^2$

$\Rightarrow 4pr = (q-4).-(q-2)$

i.e. $4pr = 8 + 2q - q^2$ $\boxed{A1}$ GIVEN ANSWER legitimately obtained 8

$v_0 = v_1 = 1$ and $v_{n+1} = -v_n^2 + 2v_n + 2$ $(p = 1, q = 2, r = 2)$

Check: LHS $= 4pr = 8$ and RHS $= 8 + 2.2 - 2^2 = 8$ also $\boxed{B1}$

Use substn. $v_n = 4u_n - 1$ $\boxed{M1}$ with $u_0 = \frac{1}{2}$

$\Rightarrow u_0 = \sin^2 \frac{\pi}{4}$ $\boxed{M1}$ giving soln. $u_n = \sin^2\left(2^n \frac{\pi}{4}\right)$ $\boxed{M1}$ Use of (i)'s result

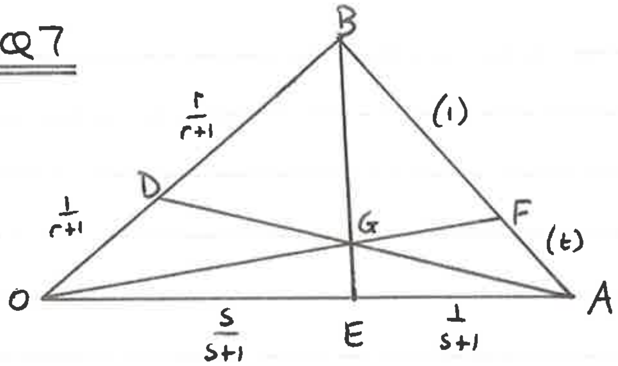
$\Rightarrow v_n = 4 \sin^2\left(2^n \frac{\pi}{4}\right) - 1$ $\boxed{A1}$

Sequence is $\{1, 3, -1, -1, -1, \ldots\}$ $\boxed{B1}$

since $v_n = -1 \ \forall n \geq 2$

6

SI 2014 Q7



$\overrightarrow{OA} = \underline{a}$
$\overrightarrow{OB} = \underline{b}$
$\overrightarrow{OG} = \underline{g}$

(i) $\underline{d} = \left(\frac{1}{r+1}\right)\underline{b}$ $\boxed{B1}$ and $\underline{e} = \left(\frac{s}{s+1}\right)\underline{a}$ $\boxed{B1}$

Eqn. AD is $\underline{r} = \underline{a} + \alpha(\underline{d} - \underline{a})$ OR $(1-\alpha)\underline{a} + \alpha\underline{d}$ $\boxed{M1}$
i.e. $\underline{r} = (1-\alpha)\underline{a} + \left(\frac{\alpha}{r+1}\right)\underline{d}$ $\boxed{A1}$

Eqn. BE is $\underline{r} = \underline{b} + \beta(\underline{e} - \underline{b})$ OR $(1-\beta)\underline{b} + \beta\underline{e}$ $\boxed{M1}$
i.e. $\underline{r} = (1-\beta)\underline{b} + \left(\frac{\beta s}{s+1}\right)\underline{a}$ $\boxed{A1}$

AD $\cap$ BE when $1-\alpha = \frac{\beta s}{s+1}$ and $1-\beta = \frac{\alpha}{r+1}$ $\boxed{M1}$ equating coeffts

$\boxed{M1}$ Solving simultaneously for α, β in terms of r, s

* See below

$\Rightarrow \alpha = \frac{1+r}{1+r+rs}$ and $\beta = \frac{r(1+s)}{1+r+rs}$ $\boxed{A1}$ $\boxed{A1}$

(OR $1-\alpha = \frac{rs}{1+r+rs}$ AND/OR $1-\beta = \frac{1}{1+r+rs}$ depending on their choices in the line eqns.)

giving $\underline{g} = \left(\frac{rs}{1+r+rs}\right)\underline{a} + \left(\frac{1}{1+r+rs}\right)\underline{b}$ $\boxed{A1}$ GIVEN ANSWER legitimately obtained

* $\boxed{M1}$ Sensible method for eliminating one of α, β e.g. $1 - \alpha + \alpha = \frac{\beta s}{s+1} + (1-\beta)(r+1)$

$\Rightarrow s+1 = \beta s + (rs + r + s + 1) - \beta(rs + r + s + 1)$

$\Rightarrow \beta = \ldots\ldots$

12

(ii) F on OG $\Rightarrow$ $\underline{f} = \lambda \underline{g}$ $\boxed{B1}$

F divides AB in ratio $t:1$ $\Rightarrow$ $\underline{f} = \left(\frac{1}{t+1}\right)\underline{a} + \left(\frac{t}{t+1}\right)\underline{b}$ $\boxed{M1}$ $\boxed{A1}$

$\boxed{M1}$ Equating terms

$$\frac{\lambda rs}{1+r+rs} = \frac{1}{t+1} \quad \text{and} \quad \frac{\lambda}{1+r+rs} = \frac{t}{t+1} \quad \boxed{A1}\ \boxed{A1}$$

Dividing (e.g.) $\Rightarrow$ $t = \frac{1}{rs}$ $\boxed{A1}$

$\boxed{M1}$

8

SI 2014 Q8

(i) L_a is $y - 0 = \left(\frac{1-a}{-a}\right)(x-a)$ $\boxed{M1}$ $\boxed{A1}$

and L_b is $y = \left(\frac{1-b}{-b}\right)(x-b)$ similarly $\boxed{B1}$ ft

Lines meet when $\left(\cancel{1}-\frac{1}{a}\right)x \cancel{+1} - a = \left(\cancel{1}-\frac{1}{b}\right)x \cancel{+1} - b$ $\boxed{M1}$

since $b \neq a$, $b - a = \left(\frac{1}{a} - \frac{1}{b}\right)x = \left(\frac{b-a}{ab}\right)x$

$\Rightarrow \underline{x = ab}$ $\boxed{A1}$

$\underline{y = (1-a)(1-b)}$ $\boxed{A1}$

6

(ii) As $b \to a$, $P \to (a^2, (1-a)^2)$ $\boxed{B1}$

i.e. $x = a^2$, $y = (1-a)^2$

Since $0 < a < 1$, $\boxed{E1}$ $a = \sqrt{x}$ $\boxed{M1}$ and $\underline{y = (1-\sqrt{x})^2}$ $\boxed{A1}$

and $0 < \sqrt{x} < 1 \Rightarrow 0 < x < 1$ $\boxed{E1}$ GIVEN ANSWER

5

(iii) $\frac{dy}{dx} = 2(1-\sqrt{x}) \cdot -\frac{1}{2}x^{-1/2} = \frac{(1-\sqrt{x})}{-\sqrt{x}}$

$\boxed{M1}$ Diffg.

$\boxed{M1}$ Good use of CHAIN RULE

$\boxed{A1}$ Correct unsimplified

3

Equ. of tgt. to C is at $(c, (1-\sqrt{c})^2)$ $y - (1-\sqrt{c})^2 = \left(\frac{1-\sqrt{c}}{-\sqrt{c}}\right)(x-c)$

$\boxed{M1}$ Attempt at tgt. provided all ~~ terms do NOT involve x

$\boxed{A1}$ correct unsimplified

$$y - 1 + 2\sqrt{c} - c = \left(\frac{1-\sqrt{c}}{-\sqrt{c}}\right)x + \sqrt{c} - c$$

$$y = \left(\frac{1-\sqrt{c}}{-\sqrt{c}}\right)x + (1-\sqrt{c}) \quad \boxed{A1}$$ Suitably simplified form

c.f. L_a (e.g.) $\quad y = \left(\frac{1-a}{-a}\right)x + (1-a) \quad \boxed{M1}$

so tgt. to C is $L_{\sqrt{c}}$ $\boxed{A1}$ where $0 < \sqrt{c} < 1$ $\boxed{E1}$

6

SI 2014 Q9

$$\left(\begin{array}{lll} \ddot{x} = -kg & \dot{x} = u\cos\theta - kgt & x = ut\cos\theta - \frac{1}{2}kgt^2 \\ \ddot{y} = -g & \dot{y} = u\sin\theta - gt & y = ut\sin\theta - \frac{1}{2}gt^2 \end{array} \right)$$

At max. ht. $\dot{y} = 0 \Rightarrow \underline{\underline{T_H = \frac{u\sin\theta}{g}}}$ $\boxed{\text{M1}}$ $\boxed{\text{A1}}$

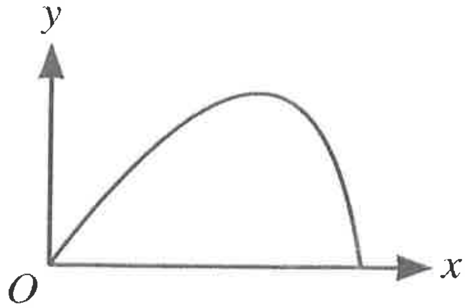
At landing, $y = 0$ $(t \neq 0) \Rightarrow \underline{\underline{T_L = \frac{2u\sin\theta}{g}}}$ $\boxed{\text{M1}}$ $\boxed{\text{A1}}$ **4**

$T = \frac{u\cos\theta}{kg}$ is the time when $\dot{x} = 0$ $\boxed{\text{B1}}$

$= \frac{u\sin\theta}{g} \cdot \frac{1}{k\tan\theta}$ $\boxed{\text{B1}}$ **2**

Case 1 $k\tan\theta < \frac{1}{2} \Rightarrow T_H < T_L < T$ $\boxed{\text{B1}}$

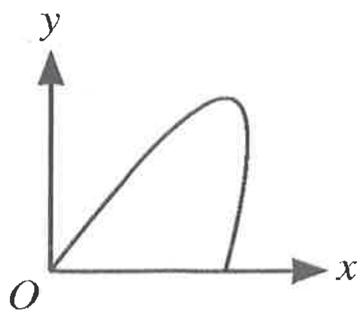
$\boxed{\text{G1}}$ Parabola-like shape



$\boxed{\text{G1}}$ "Shortened" parabola on RHS **4**

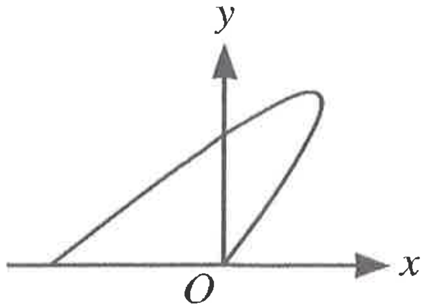
$\boxed{\text{G1}}$ Clearly never vertical

Case 2 $\frac{1}{2} < k\tan\theta < 1 \Rightarrow T_H < T < T_L$ $\boxed{\text{B1}}$



$\boxed{\text{G2}}$ **3**

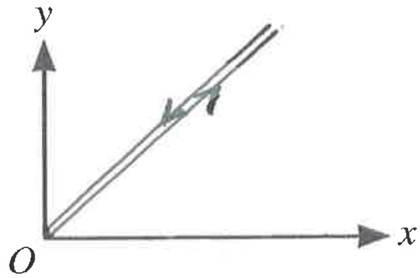
Case 3 $\quad k\tan\theta > 1 \quad \Rightarrow T < T_H < T_L$ $\boxed{B1}$



$\boxed{G2}$

3

When $k\tan\theta = 1$, $\dot{x} = \dot{y} = 0$ together $\boxed{B1}$
particle lands at O $\boxed{B1}$
$\therefore$ moves in a straight line $\boxed{B1}$ *
since resultant accln. is $\parallel$ direction of projection $\boxed{E1}$



* or $\boxed{G3}$

4

SI 2014 Q10

(i) Ball's centre of mass falls H, so $\boxed{M1}$

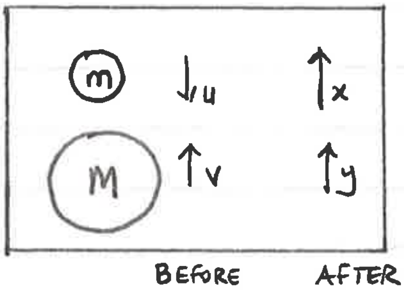
By energy (e.g.) $u^2 = 2gH$ at ground $\boxed{A1}$

NEL $v = eu$ $\boxed{B1}$

By energy (e.g.) $v^2 = 2g(H_1 - R)$ $\boxed{B1}$

Then $\underbrace{2g(H_1 - R) = e^2 \cdot 2gH}_{\boxed{M1}}$ $\Rightarrow$ $H_1 = R + e^2 H$ $\boxed{A1}$ **6**

(ii)



NB $u = \sqrt{2gH}$ and $\underbrace{v = eu}_{} = e\sqrt{2gH}$

$\boxed{B1}$ This needs to be used at some stage.

CLM ↑ $mx + My = Meu - mu$ $\boxed{M1}$ $\boxed{A1}$

NEL $x - y = e(u + eu)$ $\boxed{M1}$ $\boxed{A1}$

Solving (for x; y not needed) $\boxed{M1}$

$$mx + My = Meu - mu$$
$$Mx - My = Meu + Me^2u$$

Adding: $(m + M)x = u(Me^2 + 2Me - m)$

$$x = \frac{(Me^2 + 2Me - m)u}{M + m}$$ $\boxed{A1}$ **7**

By energy (e.g.) smaller ball's centre of mass rises a distance d, given by $\quad x^2 = 2gd$ $\boxed{\text{M1}}$

$$\Rightarrow d = \frac{1}{2g}\left(\frac{Me^2+2Me-m}{M+m}\right)^2 \cdot 2gH \qquad \boxed{\text{A1}}$$

$$\left(= H\left(\frac{M(e+1)^2}{M+m} - 1\right)^2 \right)$$

and $\quad h = 2R + r + d \qquad \boxed{\text{M1}}$

Subst$^{\underline{n}}$ $\quad R=0.2$, $r=0.05$, $H=1.8$, $h=4.5$ and $e=2/3$ $\boxed{\text{M1}}$

$$d = 1.8\left(\frac{\frac{4}{9}M+\frac{4}{3}M-m}{M+m}\right)^2 = 1.8\left(\frac{\frac{16}{9}\lambda-1}{\lambda+1}\right)^2 \qquad \lambda = \frac{M}{m} \quad \boxed{\text{M1}}$$

and $\quad 4.5 = 0.45 + 1.8\left(\frac{\frac{16}{9}\lambda-1}{\lambda+1}\right)^2$

or $\quad 10 = 1 + 4\left(\quad\right)^2$

$$\Rightarrow \frac{\frac{16}{9}\lambda-1}{\lambda+1} = \frac{3}{2} \qquad \left[\text{Condone failure to consider } -\tfrac{3}{2} \text{ (it gives } \lambda<0)\right]$$

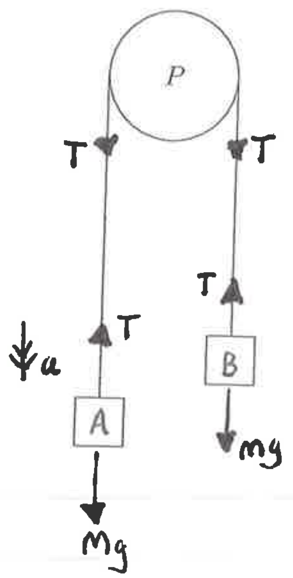
$\boxed{\text{M1}}$ Solving a suitable linear eq$^{\underline{n}}$ for λ

$$\Rightarrow \underline{\underline{\lambda = \frac{M}{m} = 9}} \qquad \boxed{\text{A1}} \quad \text{CAO}$$

7

SI 2014 Q11

(i)



N2L ↓A $\quad Mg - T = Ma$ $\boxed{B1}$

N2L ↑B $\quad T - mg = ma$ $\boxed{B1}$

Adding: $(M-m)g = (M+m)a$ $\boxed{M1}$

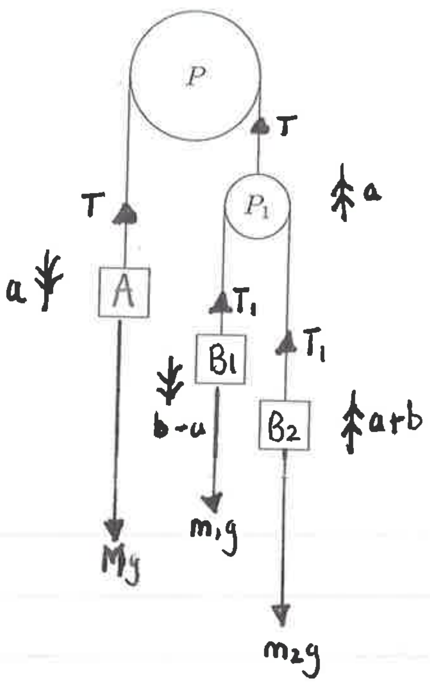
$\Rightarrow \quad a_1 = \dfrac{(M-m)g}{M+m}$ $\boxed{A1}$ GIVEN ANSWER

Subst$^{\underline{n}}$ for T and force on pulley ↓ is $2T$ $\boxed{M1}$

$\Rightarrow \quad \dfrac{4Mmg}{M+m}$ $\boxed{A1}$

6

(ii)



Let b be accln. of B_1, B_2 relative to P_1

N2L ↓A $\quad Mg - T = Ma$ $\boxed{B1}$ (1)

N2L ↓B_1 $\quad m_1 g - T_1 = m_1(b-a)$ $\boxed{B1}$ (2)

N2L ↑B_2 $\quad T_1 - m_2 g = m_2(a+b)$ $\boxed{B1}$ (3)

For P_1 $\quad T = 2T_1$ $\boxed{B1}$ (4)

+ $\boxed{B1}$ at some stage for sorting out "global" acclns. of B_1, B_2

5

e.g.

$m_2.(2)$: $\quad m_1 m_2 g - m_2 T_1 = m_1 m_2 b - m_1 m_2 a$ $\quad \boxed{M1}$ Suitable elimination (for a) process used

$m_1.(3)$: $\quad m_1 T_1 - m_1 m_2 g = m_1 m_2 a + m_1 m_2 b$

Subtracting ↑: $\quad (m_1 + m_2)T_1 - 2m_1 m_2 g = 2m_1 m_2 a$

$T = M(g-a) \Rightarrow T_1 = \frac{1}{2}M(g-a)$ $\quad \boxed{M1}$ T, T_1 eliminated

$$\Rightarrow \quad \tfrac{1}{2}M(g-a)(m_1+m_2) - 2m_1 m_2 g = 2m_1 m_2 a$$

$$\Rightarrow \quad M(m_1+m_2)g - 4m_1 m_2 g = M(m_1+m_2)a + 4m_1 m_2 a$$

$$\Rightarrow \quad a = \left(\frac{M(m_1+m_2) - 4m_1 m_2}{M(m_1+m_2) + 4m_1 m_2}\right)g$$ $\boxed{M1}$ "a" found in this form

$$\underline{\underline{a_2 = \left(\frac{M - 4\mu}{M + 4\mu}\right)g}} \quad \text{where } \mu = \frac{m_1 m_2}{m_1 + m_2}$$

$\boxed{A1}$

GIVEN ANSWER

4

$$\frac{a_1}{g} = \frac{M-m}{M+m} \quad \text{and} \quad \frac{a_2}{g} = \frac{Mm - 4m_1 m_2}{Mm + 4m_1 m_2}$$ $\boxed{M1}$

$\boxed{M1}$

$$\text{so } a_1 = a_2 \Leftrightarrow (M-m)(Mm + 4m_1 m_2) = (M+m)(Mm - 4m_1 m_2)$$

$$\Leftrightarrow \underline{M^2 m} + 4Mm_1 m_2 - Mm^2 - 4m m_1 m_2 = \underline{M^2 m} - 4Mm_1 m_2 + Mm^2 - 4mm_1 m_2$$

$$\Leftrightarrow 8Mm_1 m_2 = 2Mm^2$$

$$\Leftrightarrow 4m_1 m_2 = (m_1 + m_2)^2$$

$$\Leftrightarrow 0 = (m_1 - m_2)^2$$ $\boxed{M1}$ OR AM-GM

$$\Leftrightarrow m_1 = m_2$$

$\boxed{A1}$

$\boxed{E1}$ If they have made both directions of proof clear (clearly stated)

5

SI 2014 Q12

		Coin H		Coin T	
Die	1	$\lvert k-1\rvert$	$\frac{3}{12}$	$\lvert k-1\rvert$	$3/12$
	2	$\lvert 2k-1\rvert$	$\frac{2}{12}$	$\lvert k-2\rvert$	$2/12$
	3	$\lvert 3k-1\rvert$	$\frac{1}{12}$	$\lvert k-3\rvert$	$\frac{1}{12}$

$\boxed{B1}$ All 6 outcomes

$\boxed{B1}$ Correct accompanying probabilities

2

$$E(X^2) = \tfrac{6}{12}(k-1)^2 + \tfrac{2}{12}(2k-1)^2 + \tfrac{2}{12}(k-2)^2 + \tfrac{1}{12}(3k-1)^2 + \tfrac{1}{12}(k-3)^2$$

$\boxed{M1}$ for $\sum x^2 p(x)$

$\boxed{B1}$ using $|\ldots|^2 = (\ldots)^2$ throughout

$$= \tfrac{1}{12}\left\{6(k^2-2k+1) + 2(4k^2-4k+1) + 2(k^2-4k+4) + (9k^2-6k+1) + (k^2-6k+9)\right\}$$

$\boxed{A1}$ Correct to here

$$= \tfrac{1}{6}\left(13k^2 - 20k + 13\right)$$

$$= k + \tfrac{13}{6}(k-1)^2$$ $\boxed{A1}$ GIVEN ANSWER legitimately obtained

$\boxed{M1}$ Method for determining when this is an integer

N.B. $E(X^2) = k + 2(k-1)^2 + \frac{1}{6}(k-1)^2$

$\boxed{A1}$ so $k = 1, 7, (13, \ldots)$

6

$$E(X) = \tfrac{6}{12}|k-1| + \tfrac{2}{12}|2k-1| + \tfrac{2}{12}|k-2| + \tfrac{1}{12}|3k-1| + \tfrac{1}{12}|k-3|$$ $\boxed{B1}$

If $k=1$, $E(X) = \frac{2}{3} \notin \mathbb{N}$ $\boxed{B1}$ Shown

If $k=7$, $E(X) = 8$ $\boxed{B1}$ so $\underline{\underline{k=7}}$

3

Prob. Distn. of X (for $k=7$)

x	4	5	6	13	20
$P(X=x)$	$\frac{1}{12}$	$\frac{2}{12}$	$\frac{6}{12}$	$\frac{2}{12}$	$\frac{1}{12}$

$\boxed{B1}$ x's

$\boxed{B1}$ p's

2

$$P(X > 25) = P(20+20,\ 20+13,\ 13+20,\ 20+6,\ 6+20,\ 13+13)$$

$$= \frac{1}{144} + 2 \cdot \frac{2}{144} + 2 \cdot \frac{6}{144} + \frac{4}{144}$$

$$= \frac{21}{144}$$ $\boxed{M1}$ 6 cases and probs. multd. $\boxed{A1}$

$$P(X = 25) = P(20+5,\ 5+20)$$

$$= 2 \cdot \frac{2}{144}$$

$$= \frac{4}{144}$$ $\boxed{B1}$

$\left(P(X < 25) = \frac{119}{144}\right.$ is not required$\left.\right)$

$$E(W) = \frac{21}{144} \cdot w + \frac{4}{144} \cdot 1 + 0 = \frac{21w+4}{144}$$ $\boxed{M1}$ $E(W) = \sum w\, p(w)$ $\boxed{A1}$

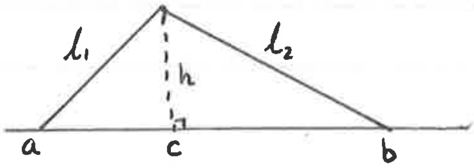
Require $\frac{21w+4}{144} - 1 < 0$ $\boxed{M1}$

$$\Rightarrow \quad w < 7$$

and max. integer value of w is $\underline{\underline{6}}$ $\boxed{A1}$

7

SI 2014 Q13



Area $= 1 = \frac{1}{2}(b-a)h \Rightarrow h = \frac{2}{b-a}$ $\boxed{B1}$

Grad. l_1 is $\frac{2}{(b-a)(c-a)}$ $\boxed{B1}$

Eqn. l_1 is $y = g(x) = \frac{2(x-a)}{(b-a)(c-a)}$ $\boxed{M1}$ $\boxed{A1}$ GIVEN ANSWER

Similarly, l_2 has eqn $y = \frac{-2(x-b)}{(b-a)(b-c)}$ $\boxed{M1}$ $\boxed{A1}$

6

(i) $$E(X) = \int_a^c \frac{2}{(b-a)(c-a)}(x^2-ax)\,dx + \int_c^b \frac{2}{(b-a)(b-c)}(bx-x^2)\,dx$$

$\boxed{M1}$ Attempted in two bits

$$= \frac{2}{(b-a)(c-a)}\left[\frac{x^3}{3} - \frac{ax^2}{2}\right]_a^c + \frac{2}{(b-a)(b-c)}\left[\frac{bx^2}{2} - \frac{x^3}{3}\right]_c^b$$

$\boxed{A1}$ $\boxed{A1}$

Including correct signs and limits

$$= \frac{2}{b-a}\left\{\frac{\frac{(c^3-a^3)}{3} - \frac{a}{2}(c^2-a^2)}{c-a} + \frac{\frac{b}{2}(b^2-c^2) - \frac{(b^3-c^3)}{3}}{b-c}\right\}$$

$\boxed{M1}$ Factorisation method at some stage

$$= \frac{2}{b-a}\left\{\frac{1}{3}(c^2+ac+a^2) - \frac{1}{2}(ac+a^2) + \frac{1}{2}(b^2+bc) - \frac{1}{3}(b^2+bc+c^2)\right\}$$ ✓ ✓

$$= \frac{2}{b-a}\left\{\frac{1}{6}(b^2-a^2) + \frac{1}{6}(bc-ac)\right\}$$ $\boxed{M1}$ $(b-a)$ factors identified

$$= \frac{1}{3}(a+b+c)$$ $\boxed{A1}$ GIVEN ANSWER

6

(ii) If $c = \frac{1}{2}(a+b)$ then $\underline{\underline{m = c}}$ $\boxed{M1}$ Case identified 2

(by symmetry) $\boxed{A1}$

If $c > \frac{1}{2}(a+b)$ then $a < m < c$ $\boxed{B1}$ Case identified

$\left(\text{OR } \frac{c-a}{b-a} > \frac{1}{2}\right)$

$$\left.\begin{aligned} &m \text{ g.b. } \tfrac{1}{2}(m-a)\,g(m) = \tfrac{1}{2} \\ &\Leftrightarrow \quad \tfrac{1}{2}(m-a)\cdot\frac{2(m-a)}{(b-a)(c-a)} = \tfrac{1}{2} \end{aligned}\right\} \boxed{M1}$$

so that $\underline{\underline{m = a + \sqrt{\frac{1}{2}(b-a)(b-c)}}}$ $\boxed{A1}$ 3

If $c < \frac{1}{2}(a+b)$ then $c < m < b$ $\boxed{B1}$ Case identified

$\left(\text{OR } \frac{c-a}{b-a} < \frac{1}{2}\right)$ and $\boxed{M1}$ Similarly*

$\underline{\underline{m = b - \sqrt{\frac{1}{2}(b-a)(b-c)}}}$ $\boxed{A1}$ 3

* May be implicit for 2nd region's working