STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 20

Warm-up

1 **(i)** Looking at the relevant areas we have:

$$\tfrac{1}{2}r \times r\tan\theta > \tfrac{1}{2}r^2\theta > \tfrac{1}{2}r^2\sin\theta\,.$$

Then we can then divide by $\frac{1}{2}$, r^2 and $\sin\theta$ (all non-zero) to get the required result.

(ii) We have $\displaystyle\lim_{\theta\to 0}\left(\frac{1}{\cos\theta}\right) = 1$ so as θ tends to 0, $\left(\dfrac{\theta}{\sin\theta}\right)$ is trapped between 1 and something that is tending to 1. Therefore we must have

$$\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = 1 \text{ and } \sin\theta \approx \theta \text{ when } \theta \approx 0\,.$$

Things to watch out for:

(1) Don't write $\sin\theta = \theta$ for small θ; it should be $\sin\theta \approx \theta$ for small θ.

(2) Don't write $\displaystyle\lim_{\theta\to 0}\left(\frac{\theta}{\sin\theta}\right) = \frac{0}{0} = 1$ because $\dfrac{0}{0}$ could be anything. For example, $\displaystyle\lim_{x\to 0}\left(\frac{x}{x^2}\right) = 0$ even though both the numerator and denominator tend to 0.

(3) Don't write $\sin\theta \to \theta$ as $\theta \to 0$. We should only have a constant on the right hand side of $\to$. You can write $\sin\theta - \theta \to 0$ as $\theta \to 0$.

(4) Note that when taking a limit, you have to replace $>$ with $\geqslant$. For example, $\theta > \sin\theta$ for $\theta > 0$, but $\theta = \sin\theta$ when $\theta = 0$.

(5) The point of this part of the question was to derive the approximation $\sin\theta \approx \theta$, valid for small θ (which means much smaller than 1; we write this as $\theta \ll 1$). We can't therefore use this result earlier in the question to obtain the $\displaystyle\lim_{\theta\to 0}\frac{\theta}{\sin\theta}$; that is why the question says firmly 'Using part (i) ...'.

(iii) Starting from:

$$\tfrac{1}{2}r \times r\tan\theta > \tfrac{1}{2}r^2\theta > \tfrac{1}{2}r^2\sin\theta$$

we can divide by $\frac{1}{2}$, r^2 and $\tan\theta$ which will sandwich $\dfrac{\theta}{\tan\theta}$ usefully. Then it is more or less the same as part (ii).

(iv) Writing $\cos\theta = (1-\sin^2\theta)^{\frac{1}{2}}$, and using the approximation $\sin\theta \approx \theta$, we can use the binomial expansion with $x = -\theta^2$ and $n = \frac{1}{2}$.

(v) You used radians at the point where you found the area of the sector OBA.

Preparation

2 **(ii)** If X is equidistant from P and Q then we have $PX = QX$, or perhaps more usefully $PX^2 = QX^2$.

There are other ways of obtaining the required equation, but you were told to consider the distances PX and QX, so this is the method you **have** to use.

(iii) If the two equations describe the same line then the coefficients will be in the same ratio, so:

$$\frac{a}{1} = \frac{4}{a} = \frac{b}{2}\,.$$

You can use the first equality to find the possible values of a and then the second equality to find the corresponding values of b.

The STEP question

3 **(i)** It may be easier to use $AP^2 = 4BP^2$. Careful algebraic manipulation should lead to the required result.

(ii) It is easier to do this question if you write both equations in expanded form, i.e. $x^2 + y^2 + 14x - 51 = 0$ and $(k^2 - 1)x^2 + \cdots$.

If these describe the same path (we should really say locus) then the **ratios** of the corresponding coefficients must be the same. Note that the corresponding coefficients are not necessarily equal, so you **cannot** conclude (for example) that $k^2 - 1 = 1$.

Using this you can get two expressions involving k^2 one of which is $k^2 - 1 = \dfrac{2a - 2k^2 b}{14}$. These expressions can be manipulated to find two different expressions for k^2 which can then be equated.

The final part requires you to expand $(a + 7)(b^2 + 51) = (b + 7)(a^2 + 51)$. You should find a common factor of $(b - a)$ which can be divided out since it is given that $a \neq b$. Make sure that when you divide by $b - a$ you write explicitly '$a \neq b$ (given)'.

Warm down

4 **(i)** There are various ways of doing this, but you have to read the question very carefully to ensure you are working out the required probability. Those who use tree diagrams often got a bit muddled.

One way is to use a list of all the possibilities (which are equally likely):

1. Pick coin (H_1, T_1) and look at side H_1.
2. Pick coin (H_1, T_1) and look at side T_1.
3. Pick coin (H_1, H_2) and look at side H_1.
4. Pick coin (H_1, H_2) and look at side H_2.
5. Pick coin (T_1, T_2) and look at side T_1.
6. Pick coin (T_1, T_2) and look at side T_2.

The question tells you that you are looking at a head, so you must have one of the cases 1, 3 or 4. Of these three cases, two of them have a head on the other side, so the probability is $\frac{2}{3}$.

This question is equivalent to the infamous Monty Hall problem about a game show; also, to the Principle of Restricted Choice in bridge (the card game).

(ii) If you know about expectation, then you can find the expected gain per game by:

$$1 \times \mathrm{P}(\text{no sixes}) + (-1) \times \mathrm{P}(\text{one six}) + (-2) \times \mathrm{P}(\text{two sixes}) + (-3) \times \mathrm{P}(\text{three sixes})\,.$$

Equivalently, you can consider what you would expect to happen over 216 games and whether you would expect to gain money or not.