# STEP CORRESPONDENCE PROJECT

# Postmortem: Assignment 27

## STEP I question

### 1 Preparation

**(i)** To start with, you need $y = xv \Rightarrow \dfrac{dy}{dx} = v + x\dfrac{dv}{dx}$.

**(ii)** **(a)** The constant of integration on each side can be combined into one constant.

**(b)** When you rearrange you should have $\displaystyle\int \frac{\cos y}{\sin y}\,dy = x + c$. Then you can use a substitution or recall the general result

$$\int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c\,. \qquad (*)$$

You should remember the modulus signs in the log, even though very often they won't make any difference.

**(iii)** You should find that the integral simplifies to $\ln\left(\frac{32}{27}\right)$.

### 2 The STEP I question

**(i)** You should end up with an integral of the form of the form $\displaystyle\int \frac{v}{1-v^2}\,dv$. You can then use $(*)$ or partial fractions (the same technique as was used in question **1(iii)**).

**(ii)** This part requires the use of partial fractions. You should be able to write the final answer in a form which does not involve fractions or logarithms.

# STEP III question

## 3 Preparation

**(i)** The general form of the chain rule for the function of a function of a function is:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{f\,g\,h}(x)\right) = \mathrm{f'g\,h}(x) \times \mathrm{g'h}(x) \times \mathrm{h'}(x)\,,$$

where, for example, $\mathrm{fgh}(x)$ means $\mathrm{f}(\mathrm{g}(\mathrm{h}(x)))$.

**(ii)** **(a)** You will probably need to use $4^{k+1} = 4 \times 4^k = 4(9M - 6k + 1)$ at some point.

**(b)** It is a good idea to write something like:

$$\text{RTP} \sum_{i=1}^{k+1} i = \tfrac{1}{4}(k+1)^2((k+1)+1)^2$$

(RTP= required to prove) before you work on the $n = k + 1$ case so that you know what you are aiming for. Note that $\displaystyle\sum_{i=1}^{k+1} i^3 = \sum_{i=1}^{k} i^3 + (k+1)^3$ and you can use the $n = k$ case to simplify this expression. It is a very good idea to take out a factor of $(k+1)^2$ before simplifying.

**(iii)** The series expansion for $\sin x$ is $x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 + \cdots$ (and is found in all good formula books). To find the value of $\sin 0.1$ you will have to work out expressions such as $0.001 \div 6$, which is a bit messy but not difficult.

## 4 The STEP III question

The first part is the extended chain rule, and you should get something like:

$$\frac{1}{x + \sqrt{x^2+1}} \times \left(1 + 2x \times \tfrac{1}{2}(x^2+1)^{-\frac{1}{2}}\right)$$

which can then be manipulated into the given answer.

For the proof by induction, you first need to show that the statement is true when $n = 0$, for which you need to find $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}$ . Then assume true for the $n = k$ case and differentiate both sides in order to reach the $n = k + 1$ case.

You can find $\mathrm{f}(0)$, $\mathrm{f}'(0)$ and $\mathrm{f}''(0)$ by simply substituting $x = 0$ into the original function and first two derivatives. Then for $\mathrm{f}'''(0)$ use the result that you proved with $n = 1$, so:

$$(x^2+1)y^{(3)} + (2 \times 1 + 1)xy^{(2)} + 1^2y^{(2)} = 0$$

and then substitute $x = 0$ in.

Note that (for example) $y^{(3)}$ corresponds to $\mathrm{f}'''(x)$ in the Maclaurin series.

# STEP Mechanics question

## 5 Preparation

**(i)** Since $\theta$ is acute you can draw a right angled triangle to help here.

**(ii)** You would expect it to be harder to pull the block up the slope than to push it down, so do check that your answers are sensible. In each case friction will oppose the motion, but in part **(b)** gravity is helping you rather than hindering.

**(iii)** Since the string is light and inextensible, and there is no friction from the pulley, the tension is the same throughout the string. You should end up with two equations of the form:

$$m_1 g - T = m_1 a \quad \text{and} \quad T - m_2 g = m_1 a$$

and then you can eliminate T.

## 6 The Mechanics question

Note that if a plane is inclined at an angle of $\theta = \arctan\left(\frac{7}{24}\right)$ then $\tan\theta = \frac{7}{24}$ and you can find $\sin\theta$ and $\cos\theta$ from this.

**(i)** Since everything is on the point of moving, friction is at the limiting value. The tension on the string is the same everywhere. You should be able to find three equations, one for each particle. Note that $Mg = 2T$ (not $Mg = T$) in this case.

**(ii)** In this case, $A$ and $B$ move up the slope and $P$ moves down. The initial accelerations are related by $a_A + a_B = 2a_P$. The answers are not particularly nice.

# STEP Probability/Statistics question

## 7 Preparation

**(i)** **(a)** Use the fact that the sum of the probabilities is 1.
**(b)** Add together the two relevant probabilities.

**(ii)** This type of situation where you keep trying something until you are successful, and then stop, is modelled by a *geometric* distribution.
**(a)** If you get a 6 on the third roll, then you did not get a 6 on either of the first two rolls (otherwise you would have stopped already).
**(b)** If you need more than five rolls to get a 6, then the first five rolls were all not 6's, so the probability is $\left(\frac{5}{6}\right)^5$.

**(iii)** An infinite geometric sequence with $r = \frac{1}{10}$.

**(iv)** This can be done via calculus, or (perhaps more simply) by completing the square.

## 8 The Probability question

The expectation is given by

$$\begin{array}{rcccccccccc} \mathrm{E}(X) & = & \mathrm{P}(X=1) & + & 2\times\mathrm{P}(X=2) & + & 3\times\mathrm{P}(X=3) & + & 4\times\mathrm{P}(X=4) & + & \cdots \\ & = & \mathrm{P}(X=1) & + & \mathrm{P}(X=2) & + & \mathrm{P}(X=3) & + & \mathrm{P}(X=4) & + & \cdots \\ & & & + & \mathrm{P}(X=2) & + & \mathrm{P}(X=3) & + & \mathrm{P}(X=4) & + & \cdots \\ & & & & & + & \mathrm{P}(X=3) & + & \mathrm{P}(X=4) & + & \cdots \\ & & & & & & & + & \mathrm{P}(X=4) & + & \cdots \end{array}$$

Then the top row is $\mathrm{P}(X \geqslant 1)$, then second row is $\mathrm{P}(X \geqslant 2)$ etc.

If $X \geqslant 4$ then the first 3 boxes were either all daddy penguins or all mummy penguins. Similarly $\mathrm{P}(X \geqslant n) = p^{n-1} + q^{n-1}$, but only for $n \geqslant 2$. It should be fairly obvious that $\mathrm{P}(X \geqslant 1) = 1$ and not $p^0 + q^0$.

Using the expression for Expectation given at the beginning of the question you have:

$$\mathrm{E}(X) = 1 + \sum_{n=2}^{\infty} \left(p^{n-1} + q^{n-1}\right) .$$

Remembering that $1 - p = q$ and $1 - q = p$, and by using $(p+q)^2 = 1$ you should be able to show the given result.

For the very last bit, use $\mathrm{E}(X) = \dfrac{1}{p(1-p)} - 1$. Note that the minimum of $\mathrm{E}(X)$ corresponds to the maximum of $p(1-p)$.