STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 32

STEP I question

1 Preparation

There are lots of websites which you could use to check your answers, such as *desmos.com* and *wolframalpha.com*. However these should be used with caution, and are best used as a final check that your algebra did not go wrong or to help work out where you went wrong if you did.

(i) Use $\sin^2 x = 1 - \cos^2 x$ and then solve the resulting quadratic in $\cos x$.

(ii) You should find that $\sin^2 x = \frac{1}{2}(1 - \cos 2x)$. You can sketch $\sin^2 x$ as a series of tranformations of $\cos x$. You could also work directly from $\sin^2 x$; a common mistake when doing this is to draw it "pointy" rather than smooth like $\cos 2x$.

(iii) Initially you will need the chain rule (twice), but for the second derivative you will need both the chain and product rules.

(iv) This is similar to the previous part. You need the result $\dfrac{\mathrm{d}}{\mathrm{d}x}\tan x = \sec^2 x$, which you should just "know" rather than have to work out each time.

(v) Here you will need $\cos\alpha = \sin(\frac{1}{2}\pi - \alpha)$. Sketch a graph to convince yourself that this is true if you have never done so before.

(vi) The identity $1 + \tan^2 x = \sec^2 x$ is another one that is worth just "knowing", though I will admit to always mentally dividing $\cos^2 x + \sin^2 x = 1$ by $\cos^2 x$ to make sure I have the sign of the 1 correct.

2 The STEP I question

(i) In a similar way to question **1(ii)** you can write $\cos^2 x$ in terms of $\cos 2x$. Don't spend a long time plotting accurate graphs: as long as the main features are clear a sketch is enough. You should be able to see that there are two intersections of the graphs in the region $0 \leqslant x \leqslant 2\pi$ and in order to find where $\mathrm{f}(x)$ is convex you will need to solve $\frac{2}{3}\cos^2 x \geqslant \sin x$. To do this first solve $\frac{2}{3}\cos^2 x = \sin x$ to find the intersections and then use your graph to find out which way the inequality goes in each region.

(ii) In this part you will have to solve $k\sec^2 x = 2\tan x$, and you should find after simplification that $\tan x = \tan\alpha$ or $\tan x = \cot\alpha$.

STEP III question

3 Preparation

(i) The gradient is positive for all x, so the graph is strictly increasing. It passes through the origin. As $x \to +\infty$ we have $y \approx \dfrac{\mathrm{e}^x}{\mathrm{e}^x}$. You should be able to show that $y < 1$ for all x. Something similar happens for negative x; in fact, as $y(-x) = -y(x)$ the function is odd and the graph will have rotational symmetry about the origin.

(ii) Here the identities $\cosh 2A \equiv 2\cosh^2 A - 1$ and $\cosh 2A \equiv 2\sinh^2 x + 1$ will be useful. If $x < 0$ then $\tanh(x/2)$ is negative but the square root is positive, so you need an extra minus sign.

(iii) Differentiate implicitly ($\sinh y \dfrac{\mathrm{d}y}{\mathrm{d}x} = 1$) and use $\sinh^2 x = \cosh^2 x - 1$.

(iv) When integrating directly, start by considering what happens when you differentiate $\sqrt{x^2+1}$.

(v) We have $x = \sin x$ when $x = 0$, and also $x > \sin x$ for $x > 0$ ('clearly', because the gradient of the graph $y = x$ is 1 and the gradient of the graph $y = \sin x$ is less than 1). We therefore have:

$$\int_0^x x\,\mathrm{d}x > \int_0^x \sin x\,\mathrm{d}x\,.$$

The final answer gives you a hint as to which functions you need to consider.

4 The STEP III question

(i) We have $\tanh(0/2) = 0$ so the two graphs pass through the origin. You can also show that the gradient of $\tanh(\frac{1}{2}y)$ is less than 1 for $y > 0$. If we set $y = \operatorname{arcosh} x$ then $\dfrac{x-1}{\sqrt{x^2-1}} = \dfrac{\cosh y - 1}{\sinh y}$ which simplifies to $\tanh(\frac{1}{2}y)$.

This part needs a little bit of bravery, but the given result can be used to help you decide what to do. Comparing the two inequalities suggests that using $y = \operatorname{arcosh} x$ might be useful.

(ii) Here you need to use:

$$\int_1^x \operatorname{arcosh} t\,\mathrm{d}t > \int_1^x \frac{t-1}{\sqrt{t^2-1}}\,\mathrm{d}t\,.$$

Both integrals can be found by using the substitution $t = \cosh\theta$. You will need to do some rearranging to get the final result.

(iii) Integrate again, using the same substitution.

STEP Mechanics question

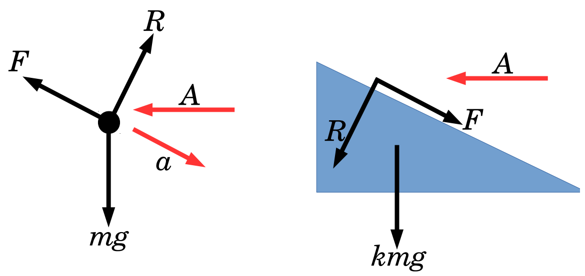
5 Preparation

(i) Draw a right-angled triangle with opposite length 3 and adjacent length 4.

6 The Mechanics question

The forces here are the weight of the particle (mg), the weight of the wedge (kmg), the friction between the particle and the wedge (F) and the normal reaction force (R) between the particle and the wedge. It is probably easiest to consider the forces on the particle first to get the correct directions of F and R and then you can draw a force diagram for the wedge, showing F and R acting in the opposite direction.

It would be a good idea to draw two seperate diagrams, one for the particle and one for the wedge, something like this (reaction of the horizontal surface on the wedge is omitted):



(i) To find the acceleration of the wedge (A in my diagram) resolve forces horizontally for the wedge and the particle. The terms involving R and F should cancel nicely.

If the particle seems to be following a line inclined at 45° then the horizontal and vertical compenents of the acceleration are the same. i.e.:

$$A - a\cos\theta = a\sin\theta\,.$$

You can substitute for A and rearrange to get the required result.

If $k = 3$ then $\tan\theta = \frac{3}{4}$. You will need to use $F = \mu R$ and eliminate R; one way is to use the vertical force equation for the particle and the horizontal force equation for the wedge. The final answer is not particlularly "nice".

(ii) The horizontal equation for the wedge is $kmA = R\sin\theta - F\cos\theta$ which can be written as $kmA = R\cos\theta(\tan\theta - \mu)$ (assuming that the particle is moving and so $F = \mu R$). Hence if $\tan\theta \leqslant \mu$ then the wedge and particle are stationary — so the answer is "nothing happens".

STEP Probability/Statistics question

7 Preparation

(i) There are a couple of ways of thinking about this. One is to add together the probability that I get the cherry bun on my first go and the probability that I pick a non-cherry and then the cherry bun.

The other way is to consider an analogous problem. Consider separating the 5 buns into a group of 2 (the ones I pick) and a group of 3 (the ones I don't pick). Then place the cherry on a bun at random — the probability that it is put on a bun in the "picked" group is $\frac{2}{5}$.

(ii) Here the simple argument is to consider the analogous problem — select r buns from the n and then place the cherry at random and the probability that the cherry is on one of the "picked" buns is $\dfrac{r}{n}$.

You can also consider the probability that the cherry bun is the first, second, third etc bun picked and using this method the probability that I pick the cherry bun is:

$$\frac{1}{n} + \frac{n-1}{n} \times \frac{1}{n-1} + \frac{n-1}{n} \times \frac{n-2}{n-1} \times \frac{1}{n-2} + \cdots$$

(iii) Here you will need to consider $\dfrac{{}^{8}C_3}{{}^{10}C_5}$ (If you have already picked the 2 males you need to choose 3 more swimmers from the 8 people left).

(iv) Lots of factors will cancel; for example, $\dfrac{(n-1)!}{n!}$ reduces to $\dfrac{1}{n}$.

(v) Here the simplest way of finding the probability is to use $\dfrac{{}^{n-2}C_r}{{}^{n}C_r}$, i.e. the number of ways of choosing r from the $n-2$ caramel chocolates divided by the total number of ways of choosing r chocolates.

(vi) When manipulating the RHS (Right Hand Side) you can pull out a factor of $\dfrac{(n-1)!}{(n-r)!r!}$ which simplifies things greatly.

The other way to think about it is to consider whether or not you include the first object when you are choosing r objects from n. The choices are:

- you **don't** choose the first object and then you have to choose r from the $n-1$ other objects
- you **do** choose the first object and then you have to choose $r-1$ from the $n-1$ other objects.

(vii) The condition $a_i/a_{i-1} > 1$ means that $a_i > a_{i-1}$ and so the sequence is increasing for $i \leqslant k$. The biggest value will be a_k but you should show convincingly that $a_{k+1} < a_k$.

8 The Probability question

(i) There are two possibilities here, either the black one is not drawn from P on the first selection or it is taken from P and is also part of the selection taken from Q and so ends up back in P. Remember that there will be $n + k$ counters in Q.

Your answer should simplify quite nicely and you should be able to see that when $k = 0$ the probability is 1. This makes sense as if you take no counters from P then the black ball stays in P but if you remove some there is a chance that the black ball will remain in Q.

(ii) Here there are three possibilities:

- the black ball is chosen from P and then neither black ball is chosen from Q (i.e. the two balls end up in Q),
- the black ball is chosen from P and then both black balls are chosen from Q (i.e. the two balls end up in P),
- the black ball is not chosen from P and the the one black ball in Q is selected (and the two balls end up in P.

When you have simplified the expression for the probability you should find that the probability that the two black balls end up in the same bag is $p_k = \dfrac{2k(n-1)}{(n+k)(n+k-1)}$.

Tlot eo maximise this you could differentiate with respect to k (pretending that it is a continuous variable), or you could investigate when $p_k/p_{k-1} > 1$. You should find that there are two values of k which maximise the probability, since there are two values of k which give the same value of p_k.