STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 13

Warm-up

1 **(i)** Careful algebraic manipulation should enable you to show that these results are true. Make sure you are secure with the index laws — in particular $a^{x+y} \neq a^x + a^y$, and $a^{2x} \neq a^{x^2}$. Also be careful to include the cross term when expanding $(a^x + a^{-x})^2$.

For $\mathrm{C}(2x)$, use $\mathrm{C}(x+y)$ replacing y with another x. Then use the result from part (a) to eliminate the $(\mathrm{S}(x))^2$.

(ii) You should have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{a^{x+h} - a^x}{h}\,.$$

You can then factorise out a^x and take it outside the limit (as a^x does not vary as h varies) to get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a^x \lim_{h\to 0} \frac{a^h - 1}{h}$$

and hence get the required result.

The second result is a bit trickier. You have

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \lim_{h\to 0} \frac{a^{-(x+h)} - a^{-x}}{h}$$

which becomes $a^{-x} \lim_{h\to 0} \dfrac{a^{-h} - 1}{h}$. Unfortunately you cannot use the given limit on this, but using a substitution of $-h = \epsilon$ (and noting that as $h \to 0$, $\epsilon \to 0$) we get

$$\frac{\mathrm{d}y}{\mathrm{d}x} = a^{-x} \lim_{\epsilon\to 0} \frac{a^{\epsilon} - 1}{-\epsilon}\,.$$

Take a negative sign outside the limit and then you can use the given result.

(iii) Using the definition in part **(i)** and the results from part **(ii)** we get:

$$\frac{\mathrm{dC}(x)}{\mathrm{d}x} = \frac{\mathrm{d}}{\mathrm{d}x}\tfrac{1}{2}\big(a^x + a^{-x}\big) = \tfrac{1}{2}\big(Ka^x - Ka^{-x}\big) = \mathrm{S}(x)$$

Then a second differentiation will get the required result.

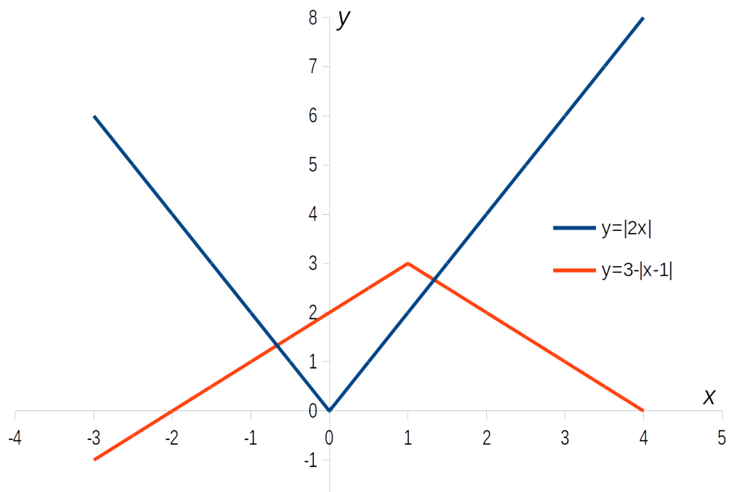
Preparation

2 **(i)** $x = -2$ or $x = 5$. It is always a good idea to substitute your answers into the *original* equation to make sure they work.

(ii) You should get a "V" shaped graph. The intercepts with the axes can be found in the usual way (i.e. substituting $x = 0$ and $y = 0$).

(iii) As the question says, there are three regions to consider. You should find that when you look at the region $0 < x < 1$ you get a solution that is out of range, so there are only two solutions.

You can sketch the graphs $y = |2x|$ and $y = 3 - |x - 1|$ to make sure your solutions are sensible. From the graph you can see that one solution is when $x < 0$ and one is when $x = 1$.



(iv) In the region $x > 0$ and $y > 0$ you need to sketch $x + y = 1$. Make sure that your line does not extend outside this region — it terminates at $(0, 1)$ and $(1, 0)$. $|x| + |y| = 1$ is symmetrical in both axes, so you can use this to sketch the graph in the region $x > 0$ and $y < 0$.

(v) It is best to start by drawing two dotted lines for $x = 1$ and $y = 1$. In the given region the equation becomes $(1 - x) + (y - 1) = 1$ i.e. $y = x + 1$. You end up with a straight line segment with end points $(0, 1)$ and $(1, 2)$.

The STEP question

3 **(i)** Start by drawing the boundary, $|x| + |y| = 1$. It might be helpful to note that there is symmetry in the x-axis and the y-axis (since $|-x| = |x|$ etc.).

Note that to be really sure that you have the correct region, you should test a point in *each quadrant*.

(ii) The easiest way to get this graph is to note that it is a translation of the previous graph with vector $\begin{pmatrix}1\\1\end{pmatrix}$.

(iii) Here you can sketch $|x| - |y| = 1$, by sketching $x - y = 1$ in the first quadrant (where x and y are both positive) and then making use of the fact that this graph will be symmetrical on both axes. Then you can apply a translation.

(iv) Here, start by sketching $xy = 1$ in the first quadrant. You can then use symmetry in the axes and a translation to get the required graph.

Note that the first two parts are clearly related. The first two parts are trying to suggest a way that you can get the second graph from the first graph. The hope is that you will then go on to use that idea for parts **(iii)** and **(iv)**.

Warm down

4 **(i)** Using the fact that tangents to a circle are equal, and letting the radius of the circle be r, we have the height of the triangle as $r + x$ and the base length is $r + y$. The area of the triangle is then $\frac{1}{2}(r + x)(r + y) = \frac{1}{2}(r^2 + rx + ry + xy)$. You also have $(r + x)^2 + (r + y)^2 = (x + y)^2$ which gives $2r^2 + 2rx + 2ry = 2xy$. Combine the two and you can express the area of the triangle in terms of x and y only.

(ii) Let $AB = 2r$ (you can actually let AB be anything you like, including 2). The small circle now has radius r, $AC = 2\sqrt{2}\,r$ and $OA = \sqrt{2}\,r$. You can then show that $\triangle AOE$ and $\triangle AEC$ are similar as they share an angle at A and $AE : AC$ equals $AO : AE$. You can then find EC and show that it equals AO.