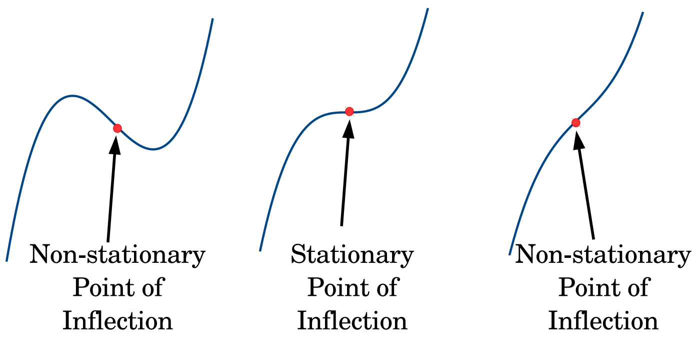
STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 14

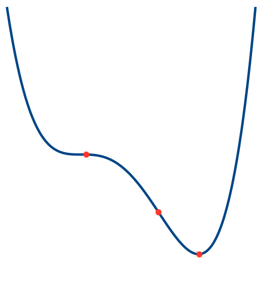
Warm-up

1 If you zoom in on the graph at a point of inflection, it looks like a straight line. For a *stationary* point of inflection the straight line is horizontal and for a *non-stationary* point of inflection the straight line is not horizontal.

A cubic graph can have a maximum and minimum (with a non-stationary point of inflection in between), or just a stationary point of inflection, or just a non-stationary point of inflection. The picture below shows the basic three cubic graph shapes (the graphs are $y = x^3 - x$, $y = x^3$ and $y = x^3 + x$).



(iii) You should find a stationary point of inflection at $(0, 0)$, a non-stationary point of inflection at $(1, -1)$ and a minimum when $x = \frac{3}{2}$. Since it is a quartic with a positive x^4 term, it will be large and positive at both "ends".



(iv) By differentiating twice, you can show that $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ when $x = 1$. You can then look at the sign of the gradient either side of $x = 1$, **or** use the fact that the graph is a translation of $y = x^4$ to show that when $x = 1$ the graph has a minimum (and not a point of inflection).

Preparation

2 You may be familiar with the idea of using intersecting graphs to see how many roots an equation has. For example, if you wanted to see how many roots $y = x^3 - 3x + 4$ has you could draw $y = -2$ onto your graph of $y = x^3 - 3x + 2$. For this question, we were asking you to use a slightly different technique, which is translating the whole graph up and down.

(iii) This is a quartic with a positive x^4 term, so will tend to $+\infty$ as x tends to $\pm\infty$. It has a stationary point of inflection when $x = -1$, a non-stationary point of inflection when $x = \frac{1}{3}$ and a minimum when $x = 1$. It looks very much like the graph in 1**(iii)**. There are no other points where the gradient is zero, or points of inflection, so make sure that you do not put any extra ones in.

If any of your graphs were incorrect, or if you just want to check them you can always plot them on a graphical calculator, spreadsheet or website such as Wolfram Alpha. Just make sure you have a go at drawing them yourself **before** you use any form of IT.

The STEP question

3 **(i)** A nice graph to draw, with y-intercept $(0, 9)$ and is shaped like a "W" (but with smooth bumps — a Mexican hat). Note that the y-intercept of $y = x^4 - 6x^2 + b$ is $(0, b)$ and that this graph is a vertical translation of the first graph.

(ii) After finding the values of a (of which there are two), it is probably easiest to sketch $y = x^4 - 6x^2 + ax$ for the positive value of a, and then you can use vertical translations of this.

For the other value of a, you can show that if the first graph is $\mathrm{f}(x)$ then the second one is $\mathrm{f}(-x)$, so is a reflection of the first graph in the y-axis. The two graphs look suspiciously like the ones you were asked to draw in 1**(iii)** and 2**(iii)**.

(iii) You can show that there are two non-stationary points of inflection. It is trickier to know how many stationary points there are (whether inflection or turning points). You can start by drawing the graph of $\frac{\mathrm{d}y}{\mathrm{d}x}$ in the case when $a = 8$, and should be able to show that there are two roots which relate to two stationary points of the original graph. You can then consider what $\frac{\mathrm{d}y}{\mathrm{d}x}$ looks like for $a > 8$ and hence show that in this case you have only one root, so the original graph has only one stationary point.

Again, you can use IT to see how accurately you have drawn your graph. You may like to draw $y = x^4 - 6x^2 + ax$ for various values of a to see what affect the ax term has.

Warm down

4 The relationship can be written as $a + 5b + 10c = b + 5c + 10a$.

(a) It is obviously true when $a = b = c$. Remember that $a = 6$.

(b) You should get a straight line with negative gradient. Since b and c have to be positive integers, the solutions lie on a finite line segment.

(c) If $(6, b, c)$ is a solution, then we have $4b + 5c = 54$. This is one equation for two unknowns, so we may expect that the solution is not unique — there are many solutions. It is only the fact that b and c are integers which prevents there being an infinite number of solutions.

Now let $b' = b + 5k$ and $c' = c - 4k$ and note that $4b' + 5c'$ is also equal to 54, i.e. b' and c' also satisfy our equation: we have found a way of generating new solutions from old!

Graphically, each solution lies on the line $4b+5c = 54$ and the coordinates of (b, c) are both integers. To find another solution we have to move along the line until we find another point on the line where both coordinates are integers. Given that the gradient of the line is $-\frac{4}{5}$ to get to another solution we have to move an integer multiple of the vector $\begin{pmatrix} 5 \\ -4 \end{pmatrix}$.

If, instead of $a = 6$, we are given that $b = 6$ then the relationship between a and c is $9a - 5c = 24$. This line has positive gradient, which means that we can find infinitely many solutions.