STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 16

Warm-up

1 **(i)** It is best to start simplifying from the bottom. You should end up with x.

(ii) The simplest way to show that $x = 15$ is a root is to set $x = 15$ in the equation and check that this gives 0. Factorising out some common factors helps with the arithmetic:

$$\begin{aligned}
&15^4 - 18 \times 15^3 + 35 \times 15^2 + 180 \times 15 - 450\\
=&15\left(15^3 - 18 \times 15^2 + 35 \times 15 + 180 - 30\right)\\
=&15^2\left(15^2 - 18 \times 15 + 35 + 12 - 2\right)\\
=&15^2\left(225 - 270 + 35 + 12 - 2\right)\ldots
\end{aligned}$$

Having established that 15 is a root of the equation, you can write

$$x^4 - 18x^3 + 35x^2 + 180x - 450 \equiv (x - 15)(x^3 - 3x^2 - 10x + 30)$$

more or less by inspection, without doing the full long division.

Alternatively, instead of substituting $x = 15$ into the equation, you could do the long division (dividing the polynomial by $(x-15)$) carefully, remembering to show that the last subtraction does indeed gives 0 (so that $x - 15$ is indeed a factor); and you should state "therefore $x = 15$ is a root" at the end — this is what you were asked to show, so you should indicate clearly that you realise that you have actually shown it and not just stopped.

(iii) For part **(a)** you should get $1 \times 2 \times 3 \times 4 = 24$.

For the other two parts write out the first few terms, then write $\cdots$ once you have seen the pattern, then write the last one or two terms. You should see that lots of things cancel.

Preparation

2 **(i)** For the convergent sequences, we were not asking you to find the limits of the sequences. However, the limits can be found by using the same techniques as in part **(iv)**. There can be a slight complication though: consider the limit of the sequence from part **(b)**:

$$\begin{aligned} l &= 6 - \frac{4}{l} \\ \Rightarrow\ l^2 - 6l + 4 &= 0 \\ \Rightarrow\ l &= 3 \pm \sqrt{5} \end{aligned}$$

There appear to be two limits! You have to do a bit of work to find out which of the two limits is the relevant one (it depends on the value of u_1). For both of the above values of l, if $u_1 = l$ then $u_2 = l$ and the whole sequence is constant, but if the value of u_1 is not equal to one of the two limits then the sequence will always tend to $3 + \sqrt{5}$: this is a *stable* limit and $3 - \sqrt{5}$ is an *unstable* limit. You will probably cover this in more detail in school (if you have not already done so).

(ii) This is a periodic sequence with period 6.

(iii) When asking you to find b when $u_3 = u_1$ we are asking you to find when the sequence has period 2, **but** this technique will also find the value of b for which $u_1 = u_2 = u_3$ i.e. the sequence is constant!

(iv) A way to find the square root of 2. Note that there are 2 possible values of l, but as $u_1 > 0$, and $u_{k+1} > 0$ if $u_k > 0$ then the whole sequence is positive and it will tend to $\sqrt{2}$ (rather than $-\sqrt{2}$).

This iterative formula was derived by rearranging $x^2 = 2$ as follows:

$$\begin{aligned} x^2 &= 2 \\ \Rightarrow\ 2x^2 &= 2 + x^2 \\ \Rightarrow\ 2x &= \frac{2}{x} + x \\ \Rightarrow\ x &= \frac{1}{2}\left(\frac{2}{x} + x\right) \end{aligned}$$

which suggests using the sequence

$$u_{n+1} = \frac{1}{2}\left(\frac{2}{u_n} + u_n\right)$$

to give successively better approximations for $\sqrt{2}$. There are other ways to rearrange the equation $x^2 = 2$ to get $x = \ldots$, some will give a sequence which converges to $\sqrt{2}$ but others will converge to $-\sqrt{2}$ and some might not converge at all.

(v) You should find that the limit is $\sqrt[3]{a}$ and so if you set $a = 7$ you will be able to find a (rather good!) approximation for $\sqrt[3]{7}$.

The STEP question

3 **(i)** In each case we start with $u_1 = 2$.

(a) Here we can put $u_2 = u_1 = 2$ and solve for k.

(b) Here we need to find an expression for u_3 in terms of u_1. Then setting $u_3 = u_1 = 2$ gives a quadratic for k. Only one of the roots of k will give a sequence which is periodic with period 2, because the other value must be the value of k found in part **(a)** which also gives $u_3 = u_1$ because it is the value of k that gives a constant sequence. It is important that your final answer gives only the one relevant value of k.

Note: We are taking the definition of *period* m to mean that the sequence repeats itself after m terms, but does not repeat itself after any number of terms smaller than m.

(c) Here we need $u_5 = u_1 = 2$, so starting by finding an expression for u_5 in terms of u_1, then set both equal to 2. This will give a quartic in k **but** you already know two of the roots (as parts **(a)** and **(b)** also have $u_5 = 2$). You can use these two roots to help you factorise your quartic. Remember to reject the two values of k which give a constant or period 2 sequence.

(ii) If $u_n \geqslant 2$ we have $u_{n+1} = 37 - \frac{36}{u_n} \geqslant 37 - \frac{36}{2} = 19$ and of course $19 \geqslant 2$. Therefore, as $u_1 \geqslant 2$, we have $u_n \geqslant 2$ for all n.

When finding the limit, you will find two possible values of l, but only one of them satisfies $l \geqslant 2$.

Warm down

4 **(i)** Creating a dump 100 miles from Cairo will solve this one.

(ii) If you can set up a dump 400 miles from the destination with enough petrol in it, you can use part (i). You therefore set up a dump 60 miles from Cairo, and make 5 one-way trips from and to Cairo to this dump.

To make a longer trip, you have to set up more frequent dumps. If the range of the car with a full tank is R, then the best strategy turns out to be having dumps at distances R, $R + R/3$, $R + R/3 + R/5$, $R + R/3 + R/5 + R/7$, etc, from the destination. Since the series

$$1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \cdots$$

(related to the so-called *harmonic* series $\sum \frac{1}{n}$) does not converge (it gets bigger without limit the more terms you add), there is no limit to the width of the dessert you can cross like this.