STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 18

Warm-up

1 You need to be quite careful with these proofs to ensure that you are not assuming something that should not be assumed.

For example, it is NOT true that $(x+y) \bmod a = x \bmod a + y \bmod a$. Take $x=9$, $y=13$ and $a=5$ to see why it doesn't hold ($2 \neq 4+3$).

(i) We have

$$N_1 + N_2 = n_1 + am_1 + n_2 + am_2$$

for some integers m_1 and m_2 and some integers n_1 and n_2 both in the interval 0 to $a-1$, and therefore.

$$(N_1 + N_2) \bmod a = (n_1 + n_2) \bmod a$$

If $n_1 + n_2 < a$ then this is equal to $n_1 + n_2$ and $n_1 + n_2 = (n_1 + n_2) \bmod a$, but if $n_1 + n_2 > a$ then we have $n_1 + n_2 = a + (n_1 + n_2) \bmod a$.

Similarly

$$N_1 N_2 = (n_1 + am_1)(n_2 + am_2)$$

and

$$(n_1 + am_1)(n_2 + am_2) = n_1 n_2 + a(m_1 + m_2 + am_1 m_2)\,.$$

Then write $n_1 n_2 = aq + (n_1 n_2) \bmod a$ for some suitable q.

(ii) Since $(10a+b) \bmod 3 = (a+b) \bmod 3$, $10a+b$ is divisible by three if and only if $a+b$ is divisible by three.

As we know $10 \bmod 3 = 1$, we have $10^n \equiv 1^n \bmod 3$, so the result holds however many digits the number has.

(iii) We note that $10^n = (11-1)^n$ so

$$10^n \equiv (-1)^n \pmod{11}$$

and

$$10^3 a + 10^2 b + 10c + d \equiv -a + b - c + d \bmod 11$$

Clearly, the same calculation works for numbers of any number of digits.

Preparation

2 **(i)** Easiest way to start here is to subtract the second equation from the first equation, and subtract the fourth equation from the third. You will then get two equations in just x and z which you can solve and then substitute into the first and third equation (for example) to get two equations in w and y.

(ii) Note that we don't have to start with $i = 1$.

(iii) Consider $\sum_{r=1}^{n} \left(\frac{1}{r} - \frac{1}{r+1} \right)$. If you write the first few terms and the last one or two you will see that lots of bits cancel. Checking you are correct by substituing a (small) value of r is always a good idea.

Note that $\sum_{r=100}^{200} = \sum_{r=1}^{200} - \sum_{r=1}^{99}$. There are some short-cuts with the arithmetic.

(iv) If two algebraic expressions are *equivalent* then they are equal in value for any value of x. Just choose some values of x.

The STEP question

3 For the first part, using $n = -1$ and $n = 1$ means that we can cancel p and r easily, and $n = 0$ will give s immediately. Once you have found p, q, r and s you need to use:

$$\sum_{r=-1}^{n} r^2 = \sum_{r=0}^{n} r^2 + (-1)^2 .$$

For the second part, the easiest thing is to take $n = -2, -1, 0, 1, 2$ — note that we need 5 equations as there are 5 unknowns (other values of n are possible but not as easy). We also need:

$$\sum_{r=-2}^{n} r^3 = \sum_{r=0}^{n} r^3 + (-1)^3 + (-2)^3 .$$

Warm down

4 **(i)** If you use $3 \equiv -2 \pmod 5$ then you get:

$$3 \times 2^{2n} + 2 \times 3^{2n} \equiv -2 \times 2^{2n} + 2 \times (-2)^{2n} \pmod 5$$

and as $2n$ is even, $(-2)^{2n} = 2^{2n}$.

(ii) Note that: $2^n + 5^n + 56 \equiv (-1)^n + (-1)^n + 2 \pmod 3$.

Note further: $2^n + 5^n + 56 \equiv 2^n + (-2)^n + 0 \pmod 7$.

And for the last part: $2^{3m} + 5^{3m} + 56 = 8^m + 125^m + 56 \equiv (-1)^m + (-1)^m + 2 \pmod 9$

(iii) Lots of ways of doing this. For example, you could start by showing that:

$$2^{3n+1} + 3 \times 5^{2n+1} = 2 \times 8^n + 15 \times 25^n$$

and then consider this modulo 17.