STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 19

Warm-up

1 **(i)** Note that $y = \dfrac{1}{x-1}$ can be considered as a translation of $y = \dfrac{1}{x}$.

(ii) If we write $\dfrac{x}{x-1} = 1 + \dfrac{1}{x-1}$ we could draw this as a translation of part **(i)**.

(iii) Here we can use long division, i.e. do $x^2 \div (x-1)$ (you may find it easier to write x^2 as $x^2 + 0x + 0$), or write $\dfrac{x^2}{x-1} = ax + b + \dfrac{c}{x-1}$ (by comparison with part (ii)), to obtain $\dfrac{x^2}{x-1} = x + 1 + \dfrac{1}{x-1}$ We can then see that as $x \to \infty$, $y \approx x + 1$, so there is an *oblique* asymptote at $y = x + 1$.

A different approach would be to sketch $y = \dfrac{(x+1)^2}{x}$, i.e. $y = x + 2 + \dfrac{1}{x}$, and then translate this to get $y = \dfrac{x^2}{x-1}$.

You can show that the graph cannot cross the oblique asymptote by trying to solve $\dfrac{x^2}{x-1} = x + 1$.

Graphs can cross oblique or horizontal asymptotes in some cases; do not regard them as impassible laser beams.

(iv) You should be able to spot the two vertical asymptotes and see that as $x \to \pm\infty$, $y \to 0$ (and for added brownie points show whether y tends to 0 from above the x-axis or below the x-axis (i.e. whether $y \to 0_+$ or $y \to 0_-$).

$y \to 0_+$ means that y tends to 0 whilst being positive (so tends to 0 "from above").

It is often useful to find the gradient. In this case, you will see that it is always negative, which helps determine the shape of the curve.

In practice situations like this you can use **Wolfram Alpha** or **Desmos** to see if you have the right idea. However, if your graph is wrong it is **vitally** important that you work out why **before** correcting it.

Preparation

2 **(i)** You can either use the quadratic formula or spot an “obvious” root and factorise to obtain x in terms of a.

(ii) If $-7x - 7 < 0$ then $x > -1$.

The consider $3x^3 - x^2 - 7x - 7 < 0$. If $x = -\frac{7}{9}$, then the first two terms are both negative (why?) and (since $x > -1$) you can state something about the last two terms.

(iii) If $a/b > 0$, then either both a and b are positive, or $\cdots$

For the second part, if $x > 0$ then to have $y > 0$ we must have $x - 1 > 0$. If $y > x$ then we have $\dfrac{x}{x-1} > x$ which can be manipulated to show that $x < 2$. At certain points in the manipulation you will use the facts that $x > 0$ and that $x - 1 > 0$. Make sure that you state these when you use them (‘since $x > 0$ we have $\ldots$’).

The STEP question

3 It may be quickest to do the whole question using the relationship:

$$\frac{1}{3}\left(b^3 - a^3\right) = \left(\frac{1}{2}\left(b^2 - a^2\right)\right)^2$$

which comes directly from doing the integrals. If you expand and divide by $(b-a)$ becomes:

$$4\left(b^2 + ab + a^2\right) = 3\left(b^3 - a^2b + ab^2 - a^3\right) .$$

When you divide by $(b-a)$ you do need to state why it is possible to do this, i.e. that since $b > a$ then $b - a > 0$.

In this last equation we can substitute $a = 0$ and $a = 1$ in order to do the first two parts of the question.

(i) Remember $b > 0$.

(ii) If you sketch the graph $y = 3x^3 - x^2 - 7x - 7$, you should be able to show that both the turning points lie below the x-axis and so there is only one root (you know the shape of a general cubic graph). Evaluating for $x = 2$ and $x = 3$ will show that the root lies between 2 and 3.

(iii) There are lots of ways of doing this. One way is to express a and b in terms of p and q and substitute these in. If you divide by q or $b - a$ you do need to state that you can do this "as $q \neq 0$".

Once you have something of the form $p^2 = \ldots$ you can use the facts that $p^2 > 0$, $q^2 > 0$ and $p \geqslant q$ (you should explain the last one) to deduce the inequalities.

Warm down

4 This question requires you to think logically and to process a lot of information.

It might help to start by thinking what would happen if they all had brown eyes — when they looked into each others eyes they would only see brown-eyed people.

(i) It is probably easiest to do part **(ii)** first — see below for a rather wordy argument.

We could also set it out like this.

We conclude, arguing by contradiction, that at least two islanders have blue eyes. Suppose not. Then exactly one islander has blue eyes, because it is given that *at least* one islander has blue eyes. But this blue-eyed islander would see no blue eyes and would deduce that it was his or her own eyes that were blue, and would leave thereby contradicting the original statement.

(ii) If one islander leaves on the boat on the first day then she[1] must have worked out that she has blue eyes. She knows that "*at least one islander has blue eyes*" so if she sees that none of the other islanders has blue eyes, then she concludes that she herself must have blue eyes and must leave. If she saw one or more islanders with blue eyes, she could not deduce anything about her own eye-colour.

For part **(i)**, we must have two or more people with blue eyes, because if exactly one person has blue eyes that person would be able to deduce that he or she has blue eyes and would leave.

(iii) If no-one leaves on the first day then we must have two or more with blue eyes. If two islanders leave on the second day they must have deduced that they have blue eyes. If they noticed that one person (other than themselves) had blue eyes on the first day, but that that person did not leave on the first day then there must be another person with blue eyes, namely themselves.

(iv) There are exactly two islanders with blue eyes.

(v) There are n islanders with blue eyes. This step requires the process of mathematical induction, though it was not expected that you would write out a formal proof. You probably said 'Well, it holds for $n = 1$, and we can use this to show that it holds for $n = 2$, and we can use this to show that it holds for $n = 3$, etc' or you might have said 'and so on' instead of 'etc'. This is the idea behind mathematical induction: you prove each step using the previous one.

[1] This is going to be impossible to write if I have to keep saying 'he or she'; let us just say 'she'.