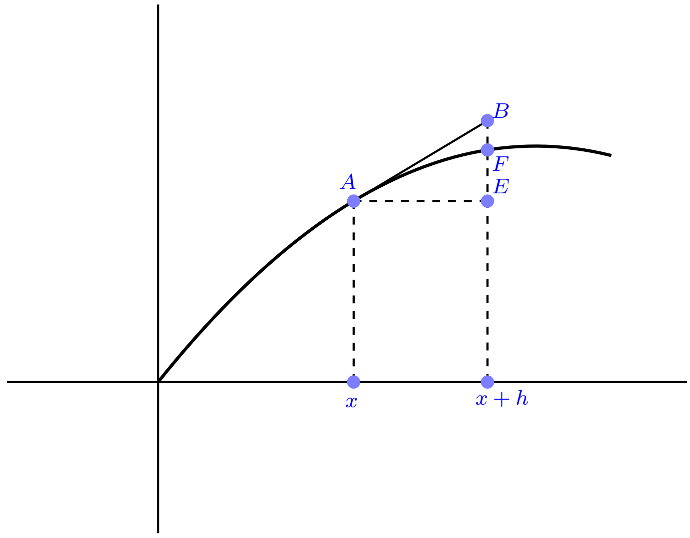
STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 22

Warm-up

1 **(i)** It doesn't much matter what $f(x)$ looks like, though a straight line would not be suitable. Note that although h is "small" *your* h should not be so small as to make your diagram cramped. Your sketch might look something like this.



A has coordinates $(x, f(x))$, and F has coordinates $(x + h, f(x + h))$. AB is tangent to the curve. You can show that B has coordinates $(x + h, f(x) + h \times f'(x))$ (e.g. by using the right-angled triangle ABE). Since h is small, B is close to F and therefore $f(h + h) \approx f(x) + hf'(x)$.

(ii) Using the definitions, and expanding

$$\big(f_1(x) + hf_1'(x)\big) \times \big(f_2(x) + hf_2'(x)\big),$$

should lead to the required result. Since the answer is given, remember to show every stage of working.

At the risk of repeating ourselves, don't assume what you are trying to prove. You may well already know the product rule, but using it in a proof of the product rule is not allowed!

2 **(i)** and **(ii)** You may already know the answer to these, and you can use your knowledge to check your answers **but** you must use the definition $(*)$ to obtain the results, as this is what the question tells you to do!

(iii) The question tells you to use the definition $(*)$, so you should **not** assume that $\mathrm{e}^{ax}\mathrm{e}^{bx} = \mathrm{e}^{(a+b)x}$, unless you can prove it using $(*)$. To do this, you would have to show that

$$\left(1 + ax + \frac{(ax)^2}{2!} + \frac{(ax)^3}{3!} + \cdots\right) \times \left(1 + bx + \frac{(bx)^2}{2!} + \frac{(bx)^3}{3!} + \cdots\right)$$
$$= \left(1 + (a+b)x + \frac{((a+b)x)^2}{2!} + \frac{((a+b)x)^3}{3!} + \cdots\right)$$

which is possible but not recommended.

For the last part (make sure you do all the bits of the question!) you can use the result just shown (with $a = 1$ and $b = -1$) to get:

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{x}\mathrm{e}^{-x}\right) = (1-1)\mathrm{e}^{x}\mathrm{e}^{-x} = 0\,.$$

This means that $\mathrm{e}^{x}\mathrm{e}^{-x} = c$ for some constant c. Substituting $x = 0$ into $(*)$ shows that $c = 1$.

This shows that $\mathrm{e}^{-x} = \dfrac{1}{e^x}$, which is again not at all obvious from $(*)$.

Preparation

3 You should not be using a calculator for any question we set (unless we specifically tell you to). It is fine to use a calculator to check your work, but we don't want to see 7.39 when you mean e^2.

(ii) You should explain why the graph passes through $(0, 1)$, ideally using $(*)$.

(iii) Using $(*)$ you can show that:

$$\mathrm{f}'(x) = 1 + 2x + \frac{3x^2}{2!} + \frac{4x^3}{3!} + \cdots\,,$$

which can be split up as:

$$\mathrm{f}'(x) = \left(1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots\right) + \left(x + x^2 + \frac{x^3}{2!} + \frac{x^4}{3!} + \cdots\right).$$

(iv)(a) and **(b)** In each of these parts you will need to state $\mathrm{e}^x \neq 0$ or $\mathrm{e}^x > 0$ at some point.

(iv) (c) To show that $x\mathrm{e}^x \to 0$ as $x \to -\infty$ you can't just substitute $-\infty$ into $(*)$!

Instead, you can use the fact that $\mathrm{e}^{-x} = \dfrac{1}{\mathrm{e}^x}$. Let $x = -t$ (where $t > 0$). Then:

$$x\mathrm{e}^x = -t\mathrm{e}^{-t} = \frac{-t}{\mathrm{e}^t}$$

Then, using $(*)$

$$\frac{-t}{\mathrm{e}^t} = \frac{-t}{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots} = \frac{-1}{\frac{1}{t} + 1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots}$$

which tends to 0 as $t \to= +\infty$.

(iv) (d) Draw $y = k$ on to your graph for some k and then consider how many intersections there are with your original graph for different values of k. The number of roots will depend on whether the line $y = k$ lies above or below the minimum, and whether it lies above or below the horizontal asymptote. Be particularly careful when it passes through the minimum point of the curve.

(v) You should find that $\mathrm{f}(x) = \sin(x^2)$ is an *even* function, i.e. it is symmetrical about the y axis. The roots occcur at $x^2 = 0, \pi, 2\pi, \cdots$ i.e. for $x = 0, \pm\sqrt{\pi}, \pm\sqrt{2\pi} \cdots$. Note that 0 is "non-negative".

The STEP question

4 **(i)** You should do the usual things: find where the graph crosses the axes; find the coordinates of the stationary points; look at the behaviour as $x \to \pm\infty$; $\cdots$. Make sure that you label all these points on your diagram, showing the relevant values.

When you have your sketch of the graph, you can consider where the line $y = k$ intersects it for various values of k.

Remember that you can use Desmos (google it if you have not heard of it) to check your sketches (when not under exam conditions!).

(ii) If the original graph is $y = \mathrm{f}(x)$ then this one is $y = \mathrm{f}(x^2)$. It will be symmetrical about the y axis, so start by considering positive x only and then reflect to get the rest of the curve.

The y intercept will remain the same, but an x intercept at $x = a$ (where $a \geqslant 0$) will move to the point $x = \sqrt{a}$. Similarly, if the graph has a stationary point at (p, q), then this will move to $(\sqrt{p}, q)$. Make sure that you label all these points on your diagram.

Warm down

5 The first three shapes all obey Euler's Theorem for *convex* polyhedra which is $F - E + V = 2$. A convex polyhedron is one for which the line segment joining any two points on the surface of the polyhedron lies inside the polyhedron. The last shape is not a convex polyhedron which means $F - E + V$ might not be 2.

You can find the number of vertices and edges of the icosahedron by considering how many faces meet at a vertex etc. Or you can sketch a net! You don't need to use Euler's theorem for this part (though you could use it).

If you take the indent all the way through (so that a hole runs all the way through the big cube) you get $F - E + V = 2$ even though this is not a convex polyhedron.