STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 23

Warm-up

1 Again, there were some people trying to use the chain rule to prove the chain rule. This is not a valid method!

(i) Note that $f'(x) = 2x$ and $g(x) = 2x^3 + 1$, so $f'(g(x)) = 2\left(2x^3 + 1\right)$. $F'(x)$ is a quintic.

(iii) For all these parts, you must make sure that you use a $\approx$ sign (rather than $=$) when you are using an approximation.
(a) $g(x+h) \approx g(x) + hg'(x)$
(b) $f\big(g(x) + t\big) \approx f\big(g(x)\big) + tf'\big(g(x)\big)$
(c) $F(x+h) = f\big(g(x+h)\big) \approx f\big(g(x) + hg'(x)\big) \approx f\big(g(x)\big) + hg'(x)f'\big(g(x)\big)$
The last step here comes from letting $t = hg'(x)$ in part **(b)**.
(d) You can either compare the result for part **(c)** to $F(x+h) \approx F(x) + hF'(x)$ or use the definition of differentiation as a limit.

(v) When doing this without the chain rule, you will find the following results useful:

$$\ln(2x) = \ln 2 + \ln x \text{ and } \ln(x^2) = 2\ln(x)\,.$$

Don't forget that $\ln 2$ is a constant and so will disappear upon differentiation.

Preparation

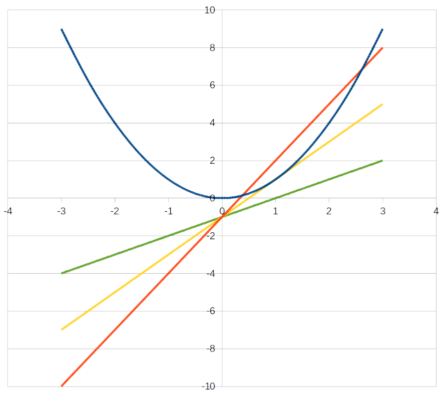
2 **(ii)** You should find that in the first case there are no intersections (the simultaneous equations have no solutions), in the second case there is one point of intersection and in the last case there are two points of intersection. Where there is one point of intersection the line is a tangent to the circle.

(iii) Where the line meets the curve, x will satisfy the equation $x^2 = ax - 1$. There will be two distinct points of intersection iff[1] $(-a)^2 - 4 \times 1 \times 1 > 0$.

Be careful when using $b^2 - 4ac > 0$; here a already appears in the equation (but not as the coefficient of x^2). Some people put the formula in quotation marks ("$b^2 - 4ac$") or you could write $b'^2 - 4a'c'$ if you wanted.

[1] iff means "if and only if", i.e. the implication works both ways. See Assignment 10 for more on this.

It is often helpful to draw a graph to get an idea of what is happening. The graph below shows $y = x^2$ and the three straight lines corresponding to $a = 1, 2, 3$. You can see that, as (positive) a increases, the line goes from not intersecting, to touching, to intersecting twice. The picture for negative a is similar, but the lines have negative gradient.



Setting $a^2 - 4 = 0$ will find the two values of a for which the straight line touches the curve.

(iv)

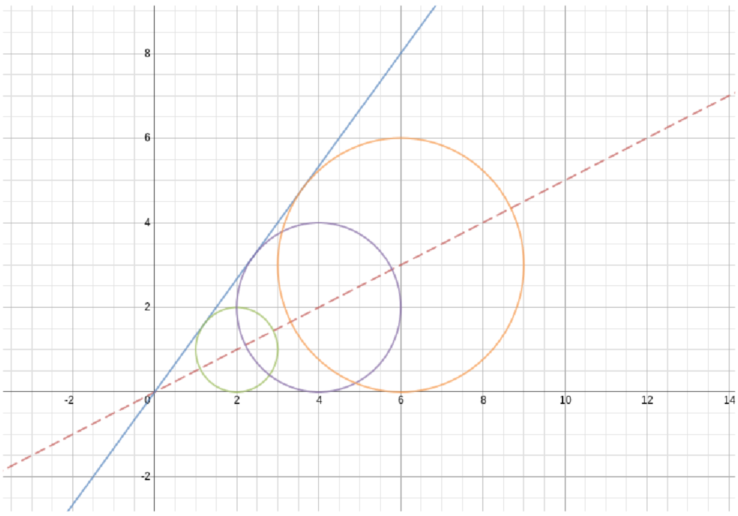
$$\tan 2\alpha = \frac{\sin 2\alpha}{\cos 2\alpha} = \frac{2\sin\alpha\cos\alpha}{\cos^2\alpha - \sin^2\alpha}\,.$$

Dividing by $\cos^2\alpha$ and using $\tan\alpha = \dfrac{\sin\alpha}{\cos\alpha}$ will lead to the required result.

The STEP question

3 **(i)** You know that this equation is a circle, radius t and centre $(2t, t)$. A nice diagram is then enough to show that the circle touches the x axis (i.e. the line $y = 0$). You could, if you prefer, substitute $y = 0$ into the first equation and show that there is only one solution, so only one point of intersection and therefore the circle touches the line.

The line joining the origin to the centre has gradient $\frac{t}{2t} = \frac{1}{2}$, so $\tan\alpha = \frac{1}{2}$. By drawing a sketch, you can see that the line that makes angle 2α with the x axis will touch the circle. The circles on the sketch below are the ones obtained by setting $t = 1, 2$ and 3, but you can sketch a general case.



If $\tan\alpha = \frac{1}{2}$ you can use the result from question **2 (iv)** to show that $\tan 2\alpha = \frac{4}{3}$ and hence the equation of this line is $y = \frac{4}{3}x$.

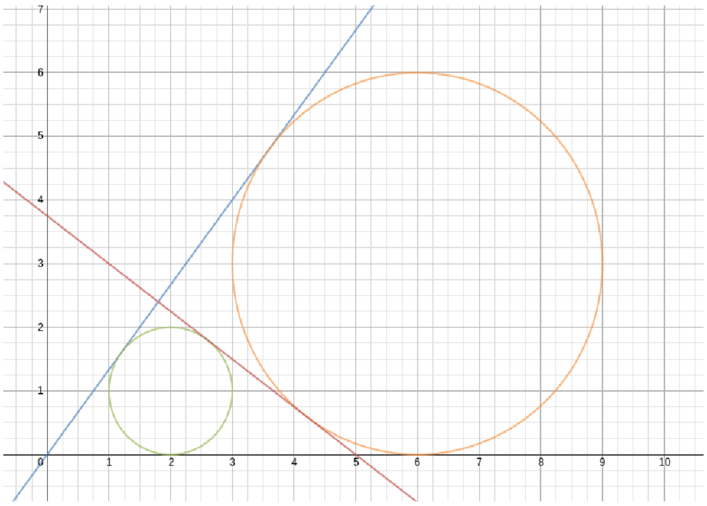
(ii) There are many ways of doing this! The following suggestion is the low-tech approach.

Since the circle required touches both the lines $y = 0$ and $3y = 4x$ then it must have the same form as in part **(i)**. The question now is: what is t?

If it *touches* $4y+3x = 15$, then there is only one solution of the simultaneous equations:

$$(x - 2t)^2 + (y - t)^2 = t^2 \quad \text{and} \quad 4y + 3x = 15\,.$$

You can then substitute for x or y. The algebra gets a bit complicated, but hang on in there. Using "$b^2 - 4ac = 0$" should lead to the equation $(t - 4)^2 = (4t^2 - 20t + 25)$. This will give two values of t, and both of these circles **touch** the three lines, but (as the picture below shows) only one of the circles lies *inside* the triangle.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Warm down

4 Fractals were very trendy about 20 years ago, but they seemed to have fallen out of the public consciousness more recently[2]. There are practical applications of fractals, including image compression, computer generated landscapes and compact antennas (produced by a company called "Fractenna").

One of the defining characteristics of fractals is *self-similarity*, i.e. if you zoom in, the small portion resembles the whole thing. If you zoom in on one side of our snowflake (actually called the *Koch Snowflake*) what you see looks like the original side.

Another characteristic of fractals is that they can have a non-integer dimension. For example, the Koch Snowflake has dimension $\log_3(4) \approx 1.2619$ — somewhere between a line and a plane, but closer to a line. Fractal dimensions have been estimated for various coastlines, and Great Britain has an estimated fractal dimension of 1.25. Others include Australia (1.13) and Norway (1.52 - it has lovely crinkly edges).

(i) The number of edges is multiplied by 4 at each stage, so the total number is 3×4^n.

(ii) The length of each edge is divided by 3 at each stage.

(iii) You should be able to show that the total length edge is a geometric progression with common ratio $\frac{4}{3}$. Hence as the number of iterations tends to infinity, the total edge length tends to infinity.

(iv) For the first iteration, the small triangles each have area $\frac{A}{9}$; try splitting the original triangle into smaller triangles each with side length $\frac{1}{3}$ of the original one to see why this is true. The area of the first iteration is therefore $A + 3 \times \left(\frac{A}{9}\right)$.

In the second iteration, each of the 12 sides will have a triangle of area $\frac{A}{9^2}$ added on, so the area is $A + 3 \times \left(\frac{A}{9}\right) + 12 \times \left(\frac{A}{9^2}\right)$.

You should find that $r = \frac{4}{9}$ and that the total area tends to $\frac{8}{5}A$. You could also find A, as this is the area of an equilateral triangle with side length 1.

One interesting conclusion is that, as $n \to \infty$, the total length of the curve tends to infinity, but the area remains bounded. This means that you would not be able to draw over all the edges of the infinite fractal, but you would be able to colour it in.

[2]Perhaps the popularity of "Frozen" will result fractals becoming well-known again.