STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 24

Warm-up

1 **(i)** You can use the website Wolfram Alpha to check your answers (only as a check though!). It can work out definite and indefinite integrals.

(ii) Part **(d)** here is a nice application of the technique of integration by parts which is not covered in all A-level courses. It does not matter which of e^x and $\sin x$ you choose to integrate for the first integration by parts, but whichever you choose you must integrate this term again on the second integration by parts. Otherwise you will end up with the true, but not very useful, statement $I = I$.

Preparation

2 **(i)** Be careful with negative signs.

(ii) You may have met cosec before, but if you have not you might like to sketch the graph of $y = \operatorname{cosec} x$. It will help to sketch $y = \sin x$ first.

(iii) You should find that most of the terms cancel leaving $\sqrt{n}$.

(iv) We were hoping for $\cos n\pi = (-1)^n$.

(v) Don't forget the constant of integration!

(vi) $I = x \sin x$ and $J = -\cos x$, so $I - J = x \sin x + \cos x$.

The STEP question

3

$$I_n - I_{n-1} = \int_0^{\pi} (\tfrac{1}{2}\pi - x)\text{cosec}(\tfrac{1}{2}x)\left(\sin(nx + \tfrac{1}{2}x) - \sin(nx - \tfrac{1}{2}x)\right)\,\mathrm{d}x\,.$$

You can then use the result from question **2 (i)** to simplify this and you should find that the cosec $(\frac{1}{2}x)$ cancels out. You will get an answer for $I_n - I_{n-1}$ in terms of n.

To evaluate I_n, note that $I_n = (I_n - I_{n-1}) + (I_{n-1} - I_{n-2}) + \cdots + (I_1 - I_0) + I_0$. You then just need to find I_0 which is a fairly simple integral.

Discussion

When n is even you should find that $I_n - I_{n-1} = 0$ as then $1 - (-1)^n = 0$. This means that you can write[1] $I_n = 4\displaystyle\sum_{i=1}^{r} \frac{1}{(2i-1)^2}$, where $r = \frac{1}{2}n$ if n is even and $r = \frac{1}{2}(n+1)$ if n is odd[2]. Then from the result

$$\sum_{i=1}^{\infty} \frac{1}{(2i-1)^2} = \tfrac{1}{8}\pi^2$$

given at the start of **(4)** it follows that $I_n \to \frac{1}{2}\pi^2$ as $n \to \infty$.

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[1]You are not asked to do this bit in the STEP question!

[2]We could write this in the form $r = \lfloor \frac{1}{2}(n+1) \rfloor$.

Warm down

4 **(i)**

$$S_{\text{even}} = \sum_{n=1}^{\infty} \frac{1}{(2n)^2} = \sum_{n=1}^{\infty} \frac{1}{4n^2} = \tfrac{1}{4} \sum_{n=1}^{\infty} \frac{1}{n^2} = \tfrac{1}{4} S\,.$$

Then $S = S_{\text{even}} + S_{\text{odd}}$ therefore $S = \frac{1}{4}S + \frac{1}{8}\pi^2$ and you can rearrange to get S.

(ii) The first sum is equivalent to $-S_{\text{odd}} + S_{\text{even}}$. For the second sum, it is probably helpful to write out some terms, such as:

$$\sum_{n=1}^{\infty} \frac{\cos \frac{1}{2}n\pi}{n^2} = \frac{0}{1^2} + \frac{-1}{2^2} + \frac{0}{3^2} + \frac{1}{4^2} + \cdots\,.$$

This sum is therefore the same as:

$$-\sum_{n=1}^{\infty} \frac{1}{(2n)^2} + 2 \times \sum_{n=1}^{\infty} \frac{1}{(4n)^2}\,.$$

(iii) Again, it might be a good idea to write down some terms. You should get:

$$\frac{1}{1^2} + \frac{1}{5^2} + \frac{1}{7^2} + \frac{1}{11^2} + \frac{1}{13^2} + \frac{1}{17^2} + \cdots\,.$$

This sum excludes the denominators which are the squares of multiples of two and three. You might think then that the required sum would be $S - S_{\text{even}} - S_{\text{multiples of 3}}$ but then the multiples of 6 would be subtracted twice. The required sum is:

$$\sum_{n=1}^{\infty} \frac{1}{n^2} - \sum_{n=1}^{\infty} \frac{1}{(2n)^2} - \sum_{n=1}^{\infty} \frac{1}{(3n)^2} + \sum_{n=1}^{\infty} \frac{1}{(6n)^2}\,.$$