STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 26

There is perhaps a little more guidance in this postmortem than in previous ones. However, we are not providing full solutions and strongly suggest that if you got stuck on a question you work through it again with the hints and guidance to help you.

If, after working through the questions again with the hints here, you are still unsure of what you should have done you can post a message in the forum for Assignment 26.

STEP I question

1 Preparation

(ii) Why was it necessary to be told $b \neq \frac{1}{2}$?

(iii) If you use the straight line form $y - y_1 = m(x - x_1)$ you should get something like $y - p^2 = -\frac{1}{2p}(x - p)$ which can then be rearranged into the required form.

(iv) The identity $a^3 - b^3 = (a-b)(a^2+ab+b^2)$ is called the difference of two cubes identity. It was discussed in an earlier assignment, and is worth remembering.

2 The STEP I question

The equation of the tangent at P is $y = -\frac{1}{2p^2}x + \frac{1}{p}$ and is very similar for Q. Solving simultaneous equations will give the required answer for T. Simplify your expression for the x coordinate before you substitute for y.

Remember that when the answer is given you must give sufficient working to justify the given answer.

In the case $pq = \frac{1}{2}$ i.e. $2pq = 1$ you should find that the x and y coordinates of T are the same, so it lies on the line $y = x$. The distance of T from the origin is $\frac{\sqrt{2}}{p+q}$.

The equation of the normal at P is $y = 2p^2x + \frac{1}{2p} - 2p^3$. You should find that the x coordinate of N satisfies the equation $2(p^2 - q^2)x = 2p^3 - 2q^3 + \frac{1}{2q} - \frac{1}{2p}$ and the difference of two cubes identity will be useful. It is also helpful to note that $p^2+pq+q^2 = (p+q)^2 - pq$ and remember that $2pq = 1$. It is probably easier to simplify your expression for the x coordinate of N before trying to find the y coordinate.

There are some hints in the discussion section of the assignment — you may want to re-visit these as well.

STEP III question

3 Preparation

(i) Completing the square on the x terms will give $(x+1)^2 + y^2 = 1$.

(ii) In both cases it is a good idea to make a sketch. For the second one it might be helpful to note that $-x^2 - \dfrac{1}{x} = -\dfrac{x^4+x}{x^2}$ and that $-x^2 - \frac{1}{x} \geqslant 0 \iff -(x^4+x) \geqslant 0$ (noting that $x \neq 0$). This true because $x^2 \geqslant 0$. Note that is would be a **bad** idea to consider the graph $-x^3 - 1$ as here we have multiplied by x which can be negative or positive!

(iii) For $y = \sqrt{x-1}$ we must restrict ourselves to $x - 1 \geqslant 0$ i.e. $x \geqslant 1$. When considering $y^2 = x - 1$ we must again have $x \geqslant 1$ but y can be negative. In fact for the third graph we have $y = \sqrt{x-1}$ **and** the reflection of this in the x axis.

4 The STEP III question

(i) Substute $z = x + iy$ into the quadratic equation and then equate the real and imaginary parts to 0. This should result in $x^2 - y^2 + px + 1 = 0$ and $(2x+p)y = 0$, as required. Note that the second of these equations implies that either $p = -2x$ or $y = 0$.

Substituting $p = -2x$ into the other equation gives the equation of a circle.

For the $y = 0$ case, you should first note that $x \neq 0$ (otherwise the first equation becomes $1 = 0$). Substituting $y = 0$ into the first equation gives the required result for p, and x can take any value except 0. This means that all the points of the form $(x, 0)$ with $x \neq 0$ lie on the root locus — they cover the real axis with the origin excluded.

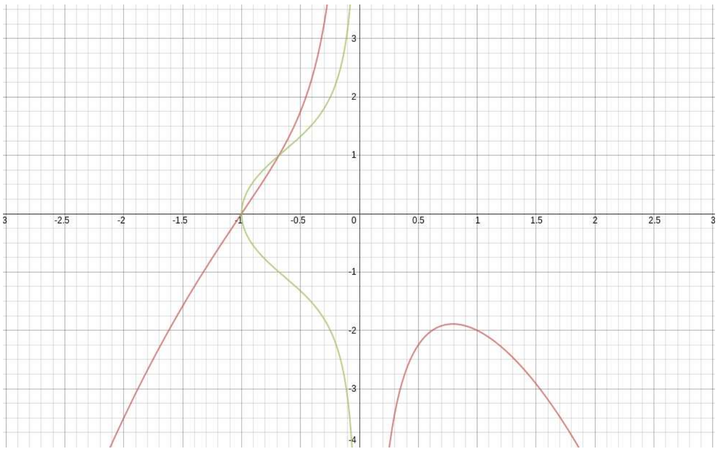
(ii) Proceed in the same way as before, and again the point $(0, 0)$ is excluded. Use the equation from the imaginary part to get two cases and then consider the other equation for each case. You should get a circle, and the real axis with a point missing again.

(iii) Start by doing exactly the same thing again, but you need to note that the real equation implies that $p \neq 0$. The imaginary equation is $py(2x+p) = 0$ and since $p \neq 0$ we have $y = 0$ or $p = -2x$.

If we take $y = 0$ we get a quadratic equation in p. We can solve for p in terms of x, but as p is real we need to discriminant to be non-negative, leading to $x^4 - 8x \geqslant 0$. This means that the points $(x, 0)$ where $x \leqslant 0$ or $x \geqslant 2$ lie on the root locus — i.e. two (infinite) sections of the real axis.

If we take $p = -2x$ we end up with $y^2 = -\dfrac{x^3+1}{x}$. You now need use the same sort of techniques that arose in the preparation. Since $y^2 \geqslant 0$ we must have $-\dfrac{x^3+1}{x} \geqslant 0$.

The graph below shows $y = -\dfrac{x^3+1}{x}$ (in red) and $y^2 = -\dfrac{x^3+1}{x}$ (in green). You can see that the second graph is restricted to $-1 \leqslant x \leqslant 0$, and also that the two graphs intersect when $y = 1$.



STEP Mechanics question

5 Preparation

(i) The expressions for horizontal and vertical displacement are:

$$x = ut\cos\phi \quad \text{and} \quad y = ut\sin\phi - \tfrac{1}{2}gt^2\,.$$

You can then solve $y = 0$ to find the time when it lands.

(ii) Substitute your value of t from part **(i)** into the expression for horizontal displacement.

(iii) You will have to differentiate $\sin\phi\cos\phi$ using the product rule. You have now shown that, if there is no air resisitance, the horizontal displacement is maximised if you fire at an angle of $\frac{\pi}{4}$ to the horizontal.

(iv) Substitute $\phi = \frac{\pi}{4}$ into the expression for distance that you found in Part (ii).

6 The Mechanics question

If the point where the cannon ball hits the slope is (X, Y) then X and Y satisfy $Y = X\tan\beta$.

If the cannon ball is fired at an angle θ to the plane (so an angle $(\theta+\beta)$ to the horizontal), you get the displacements:

$$x = ut\cos(\theta+\beta)$$
$$y = ut\sin(\theta+\beta) - \tfrac{1}{2}gt^2\,.$$

You can then substitute these into $Y = X\tan\beta$ and solve for t. It should simplify to something nice.

You could instead choose θ to be the angle with the horizontal, instead of with the plane: it won't make much difference.

You can then substitute this value of t into your expression for x. You then need to differentiate (or use some fancy trig. formulae) to find the value of θ which maximises this (remember β is fixed). You will need to use several trig identities but should be able to reduce it to a neat expression involving θ and β.

This value of θ is the angle the cannon ball makes with the inclined slope — if you want the angle the cannon ball makes with the horizontal then you need $\theta+\beta$.

You can then find the maximum horizontal distance travelled, and use trigonometry to find the distance travelled along the inclined plane (which is $\dfrac{X}{\cos\beta}$).

STEP Probability/Statistics question

Continuous Random Variable questions of this type are not usually covered until module S3 or equivalent, but as long as you don't mind a bit of integration they can be quite approachable.

7 Preparation

(i) Part **(d)** requires you to find the value M such that:

$$\int_0^M kx\,\mathrm{d}x = \int_M^2 kx\,\mathrm{d}x = \tfrac{1}{2}\,.$$

Using $\displaystyle\int_0^M kx\,\mathrm{d}x = \tfrac{1}{2}$ is probably the easiest approach to take (do the integral!).

(ii) **(a)** You need to evaluate $\displaystyle\int_0^1 x^2\,\mathrm{d}x + \int_1^{\frac{5}{3}} 1\,\mathrm{d}x$.

(b) Here you need $\displaystyle\int_{\frac{1}{2}}^1 x^2\,\mathrm{d}x + \int_1^{\frac{4}{3}} 1\,\mathrm{d}x$.

(c) Here we do $\displaystyle\int_0^1 x^3\,\mathrm{d}x + \int_1^{\frac{5}{3}} x\,\mathrm{d}x$.

(d) From what you did in part **(a)** you should have found that the first integral is less than $\frac{1}{2}$ so we have $M > 1$. It is probably easiest to use $\displaystyle\int_M^{\frac{5}{3}} 1\,\mathrm{d}x = \frac{1}{2}$.

8 The Probability question

For the first part, the sketch consists of two rectangles, the first of height a and the second height b with $a > b$. The total width of the two rectangles is 1. If $a < 1$ (and so $b < 1$) then the two rectangles will cover an area less than a 1×1 square and the total area will be less than 1. Therefore (by contradiction) $a > 1$. You can make a similar argument by considering what happens when $b > 1$.

You are not explicitly asked to do it, but you will need k in terms of a and b. For this use the fact that the total area is 1, so $ak + b(1-k) = 1$.

(i)

$$\mathrm{E}(X) = \int_0^k ax\,\mathrm{d}x + \int_k^1 bx\,\mathrm{d}x\,.$$

and you will need to use your expression for k in terms of a and b.

(ii) The crucial thing here is whether M lies in the first rectangle or the second one. If the area of the first rectangle is more than $\frac{1}{2}$ i.e. $ak > \frac{1}{2}$ then the median will satisfy $aM = \frac{1}{2}$. You then need to consider what happens if M lies in the second rectangle.

(iii) You need to consider the two values for M seperately.

If $a+b \geqslant 2ab$ then $\mathrm{E}(X) - M = \dfrac{1-2b+ab}{2(a-b)} - \dfrac{1}{2a}$. A bit of rearranging (remembering that $a > b$), you get an expression that is a complete square.

You then need to repeat for the $a+b \leqslant 2ab$ case, which also gives a complete square.