STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 28

STEP I question

1 Preparation

(i) This is a geometric series, so we can use the standard formula $S_\infty = \dfrac{a}{1-r}$.

(ii) Differentiating both sides you should find that:

$$1 + 2p + 3p^2 + \cdots = \frac{1}{(1-p)^2}\,.$$

(iii) This question is a little contrived really, but practices some of what you need for the STEP question. To find c substitute in $x = 0$.

2 The STEP I question

(i) Substitute $x = 1$. Try adding up rows of Pascal's Triangle and you will see that the sum of each row is a power of 2.

(ii) Take the expansion of $(1+x)^n$ and differentiate.

(iii) Looking at the 2^{n+1} you might think that integration is a good idea, and integrating $(1+x)^n$ is useful, but before substituting $x = 1$ you need to find the constant of integration (substituting $x = 0$ is a good plan).

(iv) You need a combination of the first and second derivatives here.

STEP III question

3 Preparation

(i) There are lots of various ways you can do these limits. As a suggestion:

(a) Use iv.
(b) Use i and ii.
(c) Use v.
(d) Use v.
(e) Use v and i.

(ii) Using $\mathrm{e}^{-t}t = \dfrac{t}{e^t}$ we have:

$$\mathrm{e}^{-t}t = \frac{t}{1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \cdots}$$

and dividing top and bottom by t gives:

$$\mathrm{e}^{-t}t = \frac{1}{\frac{1}{t} + 1 + \frac{t}{2!} + \frac{t^2}{3!} + \cdots}$$

and so as $t \to \infty$, $\mathrm{e}^{-t}t \to 0$.

For the second part, you can divide top and bottom by t^n.

(iii) If $x = \mathrm{e}^{-t}$ then as $t \to \infty$, $x \to 0$. When you substitute you will need $t = -\ln x$.

(iv) You need to write $\ln x = 1 \times \ln x$ and use integration by parts to get $\displaystyle\int_r^1 \ln x \, \mathrm{d}x = -r \ln r - (1 - r)$ and you can then consider the limit as $r \to 0$.

4 The STEP III question

For the first part use $x = \mathrm{e}^{-t}$.

(i) Using the hint, the limit becomes $\lim_{x\to 0} \mathrm{e}^{x\ln x}$ and from the first part we know that $\lim_{x\to 0} x\ln x = 0$ (taking $m = n = 1$).

(ii) If you start with I_{n+1} and integrate by parts you will get something like:

$$I_{n+1} = \left[\frac{x^{m+1}}{m+1}(\ln x)^{n+1}\right]_0^1 - k \times I_n$$

where k is something involving m and n. You will need to use the result at the beginning of the question to justify the bit in the square brackets becoming zero.

To find I_n, first note that:

$$I_n = -\frac{n}{m+1} I_{n-1} = -\frac{n}{m+1} \times -\frac{n-1}{m+1} I_{n-1} = \cdots .$$

You will have to evaluate I_0, and I_0 is **not** zero.

$$I_0 = \int_0^1 x^m \,\mathrm{d}x = \frac{1}{m+1} .$$

(iii) If you use the hint from part **(i)** you can write the integral as:

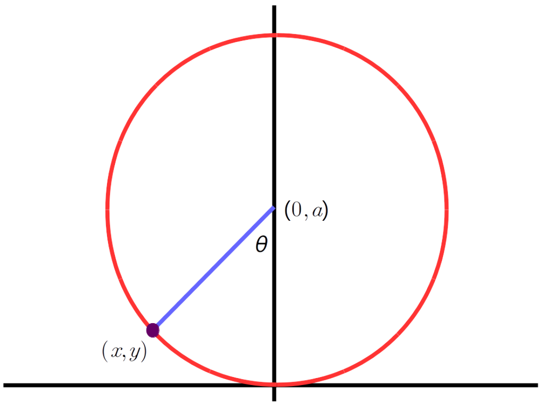
$$\int_0^1 \mathrm{e}^{x\ln x}\mathrm{d}x = \int_0^1 \left(1 + (x\ln x) + \frac{(x\ln x)^2}{2!} + \cdots\right) \mathrm{d}x$$

(using the series expansion for e^x). You can then use your result for I_n repeatedly.

STEP Mechanics question

5 Preparation

(i) Start by drawing a diagram, maybe like the one below:



You can then find expressions for the x and y co-ordinates. When differentiating with respect to time note that $\dfrac{\mathrm{d}}{\mathrm{d}t}(\sin\theta) = \cos\theta \times \dfrac{\mathrm{d}\theta}{\mathrm{d}t}$. For the speed, use Pythagoras' theorem to find the length of the velocity vector.

(ii) When the hoop has rolled through angle θ, the length of the arc which has been in contact with the floor is $a\theta$ and this is the distance the centre has moved. The y co-ordinate of the centre stays the same as the hoop rolls. The position vector of the centre is therefore $\begin{pmatrix} a\theta \\ \theta \end{pmatrix}$.

6 The Mechanics question

Here you need to combine the ideas from both parts of the previous question. The easiest way to find the coordinates of P are to consider the position of the centre and then how P is related to that. You should find that P is at:

$$\begin{pmatrix} a\theta \\ a \end{pmatrix} + \begin{pmatrix} -a\sin\theta \\ -a\cos\theta \end{pmatrix} .$$

You can then find the velocity and speed of P. Do something similar for Q and you can then find the total kinetic energy.

For the last part you can use the conservation of energy to show that $\dot{\theta}$ is constant. No forces except gravity and the vertical reaction force means that no energy is converted into heat etc.

STEP Probability/Statistics question

7 Preparation

(i) The number of arrangements is $6!$ (=720).

(ii) The number of arrangements is $10!$.

(iii) You should find that the number of ways of arranging the 10 people with the 4 men together is $4! \times 7!$. The probability is therefore $\dfrac{4! \times 7!}{10!}$ and you can cancel the $7!$ to get $\dfrac{4!}{10 \times 9 \times 8}$.

(iv) With the women together you get $\dfrac{6! \times 5!}{10!}$ which you should be able to simplify without a calculator. Note that you would expect this answer to be smaller than the previous one as to get 6 specific people together is less likely than getting 4 specific people together.

(v) Now there are 2 "rope rings", one round the women and one around the men. The total number of ways of arranging them now is $6! \times 4! \times 2$. The extra factor of 2 is because you could have the men to the left and the women to the right or vice-versa.

(vi) The total number of ways of arranging the 10 people so that no two men are together will be $6! \times 7 \times 6 \times 5 \times 4$.

(vii) This time the total number of arrangements will be $6 \times 5 \times 8!$.

8 The Probability question

There are different ways you can approach this question! The preparation leads you into a certain method, but there are other techniques you can use.

(i) If $K = 3$ then all three girls are together, and using a similar idea to before, the number of arrangements is $3! \times (n+1)!$ out of a total of $(n+3)!$.

Note that if $n = 0$, i.e. there are no boys, then the probability that all three girls stand together is 1. This could be a useful check.

(ii) Here the maximum number of girls together is 1, i.e. no two girls stand together. As in the previous question you can arrange the boys first - $n!$ ways and then slot the girls into the gaps, there being $n+1$ choices for the first girl, n for the second etc.

As the answer is given, make sure you justify your solution fully.

(iii) As $K = 1, 2, 3$ you first need to find $\mathrm{P}(K = 2)$, but this is simply a case of using the fact that the three probabilities add to 1, and as the denominators in parts **(i)** and **(ii)** are the same this is reasonably straightforward.

You would expect the expectation to be bigger than 1 but less than 3 so this might be a useful check on your answer.

9 **(i)** Find how many ways you can choose two hockey players to stand at the ends, and then work out how many ways you can arrange the rest. You can cancel a lot of factors.

(ii) Here we start by "roping" the hockey players together. Note that there are $n - r$ non-hockey players.

(iii) If there are too many hockey players, then it will be impossible to separate them. If there are $n - r$ non-hockey players then there are $n - r + 1$ gaps, so for the probability to be greater than zero we need $r \leqslant n - r + 1$. You can use factorials to write your answer neatly.