STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 29

STEP I question

1 Preparation

(i) After expanding, you should be able to write the result in the form $2N+1$.

(ii) Consider $(2a+1)+(2b+1)$.

(iii) Expanding the brackets gives $8k$. Two examples are $16 = 5^2 - 3^2$ and $40 = 11^2 - 9^2$.

(iv) In the previous part you have shown that the difference between any two *consecutive* odd squares is a multiple of 8 — now you need to show that the difference between *any* two odd squares is a multiple of 8. The given result shows that the difference is a multiple of 4, so you need to find another factor of two, by following the hint in the question.

(v) Your conjecture might be that the difference of any two even squares is a multiple of 4.

(vi) Use the formula for the difference of two squares.

(vii) Work systematically, starting (for example) with $a = 12$ (a cannot be greater than 12).

2 The STEP I question

(i) Note the stipulation of "non-zero squares". This means that you need to write 16 as $5^2 - 3^2$ rather than as $4^2 - 0^2$.

(ii) Consider $(k+1)^2 - k^2$.

(iii) Consider $(k+1)^2 - (k-1)^2$.

(iv) Consider $a^2 - b^2$ in the 4 cases depending on whether a and b are odd or even. Note that a multiple of 4 cannot be written in the form $4k+2$ (which is a multiple of 2 but not a multiple of 4).

(v) If pq can be written as the difference of two square then $pq = a^2 - b^2 = (a+b)(a-b)$. There are only two ways to factorise pq i.e. $p \times q$ and $pq \times 1$. You can then set, for example, $p = a+b$ and $q = a-b$ and solve for a and b in terms of p and q. If $q = 2$ then a and b are **not** integers.

(vi) The key here is to write $675 = 3^3 \times 5^2$ and consider the different options for $(a+b)$ and $(a-b)$.

STEP III question

3 Preparation

(i) Use $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^x = \mathrm{e}^x$ and $\dfrac{\mathrm{d}}{\mathrm{d}x}\mathrm{e}^{-x} = -\mathrm{e}^{-x}$.

(ii) Run through the usual steps for sketching a curve, and note that $\sinh x$ is an odd function, i.e. $\mathrm{f}(-x) = -\mathrm{f}(x)$ and so it will have rotational symmetry about the origin. Also $\cosh x$ is an even function and will have reflection symmetry across the y-axis. You can check your answers by using Wolfram Alpha or Desmos.

(iii) You should get something simple for each answer.

(iv) You could just multiply the two results from the previous part, or start from scratch.

(v) Separate the variables to get

$$\int \frac{\cosh y}{\sinh y}\mathrm{d}y = \int 1\mathrm{d}x\,.$$

4 The STEP III question

(i) It will be helpful to note that $\sqrt{\sinh^2 x + 1} = \cosh x$. You should get two nice answers for u. You can then use these when solving the differential equation.

The condition $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ will enable you to reject one of the two solutions.

(ii) Start by solving the quadratic equation to find two solutions for $\dfrac{\mathrm{d}y}{\mathrm{d}x}$. The standard result $\displaystyle\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\mathrm{d}x = \ln|\mathrm{f}(x)| + c$ will be useful. You can reject one solution using the condition $y = 0$ at $x = 0$.

Remember that $\cosh y \geqslant 1$; this should tell you that $x \leqslant 0$. As $x \to -\infty$, one possibility (of two) is that $y \to \infty$ in which case $\cosh y \approx \frac{1}{2}\mathrm{e}^y$.

STEP Mechanics question

5 Preparation

(i) Assume that the first particle is travelling to the right, and call this the positive direction. You should find that:

$$e = \frac{v_B - v_A}{3 - (-5)}\,.$$

(ii) Conservation of linear momentum gives:

$$5 \times 3 + 2 \times (-5) = 5v_A + 2v_B\,.$$

It is always a good idea to make sure that your answers make sense. If you end up with A still travelling to the right and B travelling to the left then this means that they have passed "through" each other in which case you have probably made a sign error in the coefficient of restitution equation.

(iii) You should get:

$$\left(\tfrac{1}{2} \times 3^2 + \tfrac{1}{2} \times 5^2\right) - \left(\tfrac{1}{2}v_A^2 + \tfrac{1}{2}v_B^2\right)\,.$$

It's a bit messy as the final velocities have denominators of 7.

6 The Mechanics question

You could call the initial velocities u_1, u_2 and u_3 (so that $u_1 = v, u_2 = 0, u_3 = 0$), the middle velocities v_1, v_2 and v_3, and the final velocities w_1, w_2 and w_3. You can then find 4 equations connecting $m_1, m_2, m_3, e, e', v, u_2$ and w_2. Three of these equations can be used to write e' in terms of the masses, as required.

The inequalities $0 \leqslant e' \leqslant 1$ imply $0 \leqslant m_2 + m_3 - m_1$ and $m_2 + m_3 - m_1 \leqslant m_1$.

The final energy of the system is $\frac{1}{2}m_2w_2^2 + \frac{1}{2}m_3v^2$. You can use your previous equations to write w_2 in terms of v and end up with an expression involving m_1, m_2, m_3 and v only. Of these, all are fixed except m_2. The maximum value m_2 can take is $2m_1 - m_3$ and the minimum value is $m_1 - m_3$ (which come from $2m_1 \geqslant m_2 + m_3 \geqslant m_1$).

STEP Probability/Statistics question

7 Preparation

(i) Let $3^x = y$, and solve the resulting quadratic. There is only one (real) solution for x. (The equation in the question was supposed to be $9^x - 3^x - 2 = 0$, which explains the hint in the question. (Whoops!))

(ii) Rearrange to get $1 - x > \sqrt{1-x}$. The right hand side is positive, so the left hand side must be positive, which means that you can square both sides without changing the direction of the inquality. Squaring both sides can introduce extra solutions, so you should **always** check that your answers satisfy the original equation.

(iii) For part **(b)** find $1 - \mathrm{P}(\mathrm{X} = 0)$.

(iv) The values of the means of the Poissons in the four parts are: 8, 16, 4 and 2.

(v) The total number of orders follows a Poisson distribution with mean 7.

8 The Probability question

In the first hour George receives no texts, and in the next hour he receives at least 1 text. The two hours are independent and the number of texts in each one follows a Poisson distribution with mean λ. Therefore $p = \mathrm{e}^{-\lambda} \times (1 - \mathrm{e}^{-\lambda})$, which gives the required quadratic equation.

The condition $4p < 1$ is required in order for the values of e^{λ} to be real. You need to show that $\mathrm{e}^{\lambda} > 1$ for each root (so that $\lambda > 0$). This involves showing that $1 + \sqrt{1-4p} > 2p$ (which is true as the LHS is greater than 1 and the RHS is less than $\frac{1}{2}$) and $1 - \sqrt{1-4p} > 2p$ (which can be shown by rearranging and squaring).

Each of Mildred's phones satisfies the same conditions as George's phone. This means that λ_1 and λ_2 are the two roots of the equation $p\mathrm{e}^{2\lambda} - \mathrm{e}^{\lambda} + 1 = 0$ that you derived in the first part of the question. You can then show that $\mathrm{e}^{\lambda_1} \times \mathrm{e}^{\lambda_2} = \mathrm{e}^{\lambda_1 + \lambda_2} = \frac{1}{p}$.

The total number of texts that Mildred receives is modelled by a Poisson distribution with mean $\Lambda = \lambda_1 + \lambda_2$. Similarly to the first part of this question, the probability that she waits between 1 and 2 hours to receive her first text (which can appear on either phone) is $\mathrm{e}^{-\Lambda} \times (1 - \mathrm{e}^{-\Lambda})$.