STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 30

STEP I question

1 Preparation

(i) If you use a substitution such as $\phi = 3\theta$ then remember that you will need to look for values of ϕ in the range $0 \leqslant \phi < 6\pi$.

(ii) A sketch will help here. One solution is in the range $\frac{1}{2}\pi < \theta < \pi$. The other solutions will be related to the first solution by $\pm 2n\pi$ (use your sketch to convince you of this). The question wanted you to find all the solutions.

(iii) Start by considering $\sin(2\theta + \theta)$. Once you have $\sin 3\theta$, you can consider $\sin 3\left(\frac{1}{2}\pi - \theta\right) = \sin\left(\frac{3}{2}\pi - 3\theta\right)$ in order to find $\cos 3\theta$.

(iv) If you set $x = -\dfrac{1}{z}$ in the first equation, you will get the second equation. This means that the roots of the second equation are given by $-\dfrac{1}{\alpha}$, where α is a root of the first equation. We were not expecting you to notice this, though you might have spotted it after you found the roots.

(v) Split this inequality into its two parts. For one of the inequalities you have to show that $-1 < 8 - 5\sqrt{3}$, which can be done by considering the equivalent $5\sqrt{3} < 9$ and squaring both sides (which is fine as both sides are positive).

2 The STEP I question

(i) First find $\sin\theta$ which can be done by using $\cos^2\theta + \sin^2\theta \equiv 1$ and by noting that $\sin\theta < 1$ in the given interval for θ. When you have found an expression for $\cos 3\theta$, it might be a good idea to check it by substituting a value of θ, such as $\frac{1}{6}\pi$ or $\frac{1}{3}\pi$.

(ii) Start by using $\tan 3\theta = \dfrac{\sin 3\theta}{\cos 3\theta}$. The cubic equation has one rational root, so you can factorise out a linear factor. If $\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{2}$ then we have $1 \leqslant \tan\theta < \infty$, and only one of your values of $\tan\theta$ will lie in this range.

STEP III question

3 Preparation

(i) **(a)** When $n = 1$, the conjecture is trivially true. For the induction step, note that $(\cos\theta + i\sin\theta)^{k+1} = (\cos\theta + i\sin\theta)^k \times (\cos\theta + i\sin\theta)$. You can then use the $n = k$ result and then multiply out the two brackets.

(b) We have $r^5(\cos 5\theta + i\sin 5\theta) = 1$. The imaginary part of this equation implies that $\sin 5\theta = 0$, and hence $\cos 5\theta = \pm 1$, but as $r \geqslant 0$ we must have $\cos 5\theta = 1$. This gives 5 possible values of θ, and $r = 1$

(c) Write $(\cos\theta + i\sin\theta)^5 = \cos 5\theta + i\sin 5\theta$. Then expand the left hand side and equate real parts.

(ii) Start with $(x - \alpha_1)(x - \alpha_2)\cdots(x - \alpha_n) = x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n$. When you imagine multiplying out the brackets you find that the term independent of x is $(-1)^n\alpha_1 \times \alpha_2 \times \cdots \times \alpha_n$.

4 The STEP III question

If we write $z = \cos\theta + i\sin\theta$, then $z^7 = 1 \implies \cos 7\theta = 1$ and $\theta = -\frac{6\pi}{7}, -\frac{4\pi}{7}, -\frac{2\pi}{7}, 0, \frac{2\pi}{7}, \frac{4\pi}{7}$ and $\frac{6\pi}{7}$.

Considering instead $(\cos\theta + i\sin\theta)^7 = 1$ the imaginary equation will give
$7c^6s - 35c^4s^3 + 21c^2s^5 - s^7 = 0$ (where $c = \cos\theta$ and $s = \sin\theta$). One solution is $s = 0$, otherwise we can divide both sides by c^6s (why can we divide by c?) to get something like the required equation.

The product of the 6 roots of the given equation is -7, and you know that the roots are of the form $\tan\theta$ for the values of θ shown above (excluding $\theta = 0$ as this corresponds to $\sin\theta = 0$). It will also be helpful to note that $\tan(-\theta) = -\tan\theta$.

For the last two cases, follow the same steps again (but you don't actually need to find the entire equation in t). What would be different if you were asked to do this for $n = 8$?

The solutions to the equation $z^n = 1$ are called the n^{th} roots of unity. They are all positioned on a unit circle, centre the origin, and the n of them are equally spaced out around this circle.

STEP Mechanics question

5 Preparation

(i) The accelerations are $\ddot{x} = \dfrac{F}{m}$ and $\ddot{y} = 0$, the initial velocities are $\dot{x} = u\cos\alpha$ and $\dot{y} = u\sin\alpha$, and the particle starts with $x = y = 0$. To find the required times, you need to:
(a) set $x = 0$, and solve for t
(b) set $x = 0$ as before,
(c) set $y = x$ and solve for t.

(ii) Note that $R\sin(2\theta)\cos\beta + R\cos(2\theta)\sin\beta = g\sin(2\theta) + f\cos(2\theta)$. This gives us the simultaneous equations $g = R\cos\beta$ and $f = R\sin\theta$. Squaring and adding will give R in terms of f and g, and the maximum value will be equal to this. Dividing one equation by the other will eliminate R and give $\tan\beta$ in terms of f and g.

6 The Mechanics question

In this question, $m = 1$ which makes things nicer! The accelerations are $\ddot{x} = -f$ and $\ddot{y} = -g$, and you can integrate twice as before to find x and y. To find OA, first find the positive value of t (let this be T) for which $y = 0$ and then substitute this into x.

(i) If the projectile starts to go back to O then we will have $\dot{x} < 0$ for some $t < T$. Substitute $t = T$ into $\dot{x} < 0$.

(ii) The distance OB can be found by substituting $\theta = 45°$ into the expression for OA. Your expression for OA can be written in the form $k(g\sin 2\theta + f\cos 2\theta) - c$, and by using the same idea as in question **5(ii)** you can find the maximum value of OA.

If $f = g$ and $\theta = 45°$ the equations for x and y are the same, so the particle moves on the line $y = x$ and lands back at the origin.

STEP Probability/Statistics question

7 Preparation

(i) The answers here are **(a)** $\frac{1}{5}$ and **(b)** $\frac{1}{4}$.

(ii) There are 24 possible different numbers here. Whilst there are more elegant methods, the simplest way to solve this question is to simply write out all 24. If one of these numbers is divisible by 4 then the last two digits must be one of "12", "24" or "32".

8 The Probability question

(i) As in question **7(ii)**, the easiest way here is probably to list all the options.

(ii) There are various was of thinking about this. One way is to look at which ice-cream is first. If the biggest ice-cream (1) is first (which happens with probability $\frac{1}{n}$), then the probability that I pick the biggest ice-cream is 0. If the second biggest ice-cream (2) is first then the probability that I pick the biggest is 1. If the third biggest ice-cream (3) is first, then I will pick the biggest one as long as 1 comes before 2. We can ignore the positions of all the other ice-creams, so the relative positions of the two ice-creams are $1, 2$ and $2, 1$. In the first case I will take the biggest ice-cream so the probability is $\frac{1}{2}$.

Putting this all together gives:

$$P_n(1) = \frac{1}{n} \times 0 + \frac{1}{n} \times 1 + \frac{1}{n} \times \frac{1}{2} + \cdots .$$

Think about what happens if the fourth biggest ice-cream is first. Then in order to pick the biggest ice-cream you need 1 to occur first out of $1, 2, 3$. You should be able to express $P_n(1)$ nicely using summation convention.