STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 31

STEP I question

1 Preparation

(i) **(a)** Use the fact that $\overrightarrow{AB} = \overrightarrow{AO} + \overrightarrow{OB}$.

(b) Here we have $\overrightarrow{OC} = \overrightarrow{OA} + \frac{2}{3}\overrightarrow{AB}$.

(c) This point lies on the line AB or AB produced (i.e. extended). If $0 < \lambda < 1$ then the point will lie between A and B, splitting it in the ratio $AX : XB = \lambda : 1 - \lambda$.

(ii) **(a)** This operation is not *commutative*, which means that, in general, $a \oplus b \neq b \oplus a$.

(b) Here you need to solve $ab - a = ba - b$. Note that $ab = ba$ as this is just normal multiplication, so you can subtract ab from both sides (normal subtraction!).

(c) The operation is also not *associative* as in general $a \oplus (b \oplus c) \neq (a \oplus b) \oplus c$.

(d) You need to find the conditions for which $a(bc - b) - a \neq (ab - a)c - (ab - a)$. You should end up with a condition on a and a condition on c.

2 The STEP I question

In this question, $X * Y$ is the point on the line XY which divides the line in the ratio $\lambda : 1 - \lambda$. You don't need to use this geometrical interpretation for most of the question, but it can help with a geometrical argument for part **(iv)**.

(i) If you think about what $X * Y$ and $Y * X$ represent you should be able to see why the answer must be $\lambda = \frac{1}{2}$. When solving $\lambda\mathbf{x} + (1 - \lambda)\mathbf{y} = \lambda\mathbf{y} + (1 - \lambda)\mathbf{x}$ you will need to compare coefficients of $\mathbf{x} - \mathbf{y}$, which is fine as $\mathbf{x} \neq \mathbf{y}$, but you do need to state this when you use it.

(ii) Here you should get two conditions on λ and a condition connecting two of $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$.

(iii) Notice that in the first case Z is outside the bracket on the LHS and appears twice on the RHS. For the second result, by comparison with the first result, you might expect X to appear twice on the RHS. You have to expand your second result to check that it does indeed hold.

(iv) You should be able to show that $P_n = \lambda^n \mathbf{x} + (1-\lambda^n)\mathbf{y}$, either by using a geometrical argument for the positions of $P_1, P_2, \ldots$ or by using and induction argument. You can then say what ratio P_n splits the line XY into.

STEP III question

3 Preparation

(i) This bit is quite hard; note that in the STEP question you were not asked to show why the corresponding result was true. You could argue that if you rotate the triangle through $\frac{2\pi}{3}$ about the origin, then the vector sum will still be the same, as $\mathbf{p}_2$ is now where $\mathbf{p}_1$ was etc. The picture is the same but the labels are different. The only point that remains in the same place after the rotation is the origin.

(ii) The three points lie on a unit circle, so $\mathbf{p}_i.\mathbf{p}_i = 1$ for each i. We also have $(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3).\mathbf{p}_1 = \mathbf{0}.\mathbf{p}_1 = 0 \implies \mathbf{p}_2.\mathbf{p}_1 + \mathbf{p}_3.\mathbf{p}_1 = -1$. By symmetry $\mathbf{p}_2.\mathbf{p}_1 = \mathbf{p}_3.\mathbf{p}_1$.

(iii) You need to use $(\mathbf{x} - \mathbf{p}_1).(\mathbf{x} - \mathbf{p}_1) = \mathbf{x}.\mathbf{x} - 2\mathbf{x}.\mathbf{p}_1 + \mathbf{p}_1.\mathbf{p}_1$. Then

$$\sum_{i=i}^{3}(k^2 - 2\mathbf{x}.\mathbf{p}_i + 1) = 3k^2 - 2 \times \mathbf{x}.(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3) + 3.$$

(iv) If you let $\mathbf{p}_2 = (a, b)$ then you can use $\mathbf{p}_2.\mathbf{p}_1 = -\frac{1}{2}$ to find b and $\mathbf{p}_2.\mathbf{p}_2 = 1$ to find two values for a (and so find $\mathbf{p}_3$ at the same time).

4 The STEP III question

Here you use $\mathbf{p}_1.(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) = 0$ along with $\mathbf{p}_1.\mathbf{p}_1 = 1$ and a symmetry argument.

(i) In this case, $\mathbf{x}$ lies on the sphere, so $\mathbf{x}.\mathbf{x} = 1$. Similarly to question **3(iii)** we have $\mathbf{x}.(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4) = 0$.

(ii) Start by using $\mathbf{p}_1.\mathbf{p}_2 = -\frac{1}{3}$ to get b and then use $\mathbf{p}_2.\mathbf{p}_2 = 1$. Then let $\mathbf{p}_3 = (p, q, -\frac{1}{3})$, say, and use $\mathbf{p}_2.\mathbf{p}_3 = -\frac{1}{3}$ and then $\mathbf{p}_3.\mathbf{p}_3 = 1$.

(iii) First note that $(XP_i)^4 = ((\mathbf{p}_i - \mathbf{x}).(\mathbf{p}_i - \mathbf{x}))^2$ and then that $(\mathbf{p}_i - \mathbf{x}).(\mathbf{p}_i - \mathbf{x}) = 2 - 2\mathbf{p}_i.\mathbf{x}$. It will probably also help to note that $(1 - \mathbf{p}_i.\mathbf{x})^2 = 1 + (\mathbf{p}_i.\mathbf{x})^2 - 2\mathbf{p}_i.\mathbf{x}$ and so

$$\sum_{i=1}^{4}(XP_i)^4 = 16 + 4\sum_{i=1}^{4}(\mathbf{p}_i.\mathbf{x})^2 .$$

All that is left to do now is expansion and simplification of brackets. It is not difficult, but care is needed to avoid making mistakes on the way. If everything goes well you should find (remember $x^2 + y^2 + z^2 = 1$) that $\displaystyle\sum_{i=1}^{4}(XP_i)^4 = \frac{64}{3}$.

STEP Mechanics question

5 Preparation

The centre of gravity is the weighted mean position of the centres of gravity of the different parts of the whole structure, i.e.

$$\bar{\mathbf{x}} = \frac{\sum m_i \mathbf{x}_i}{\sum m_i}.$$

For both this question, and question **6** the rods are identical, so the masses are the same and the centre of gravity is simply the mean position of the separate centres of gravity of the individual rods.

(i) Here the centre of mass is given by:

$$\frac{(0,a) + (a,0)}{2}$$

(ii) A carefully drawn diagram should show that $\tan BGX = 1$ and $\tan AGX = 3$.

(iii) You should be able to see that $\alpha = BGX + AGX$. So we have $\tan\alpha = \tan(\tan^{-1} 1 + \tan^{-1} 3)$ and you can use the formula for $\tan(A + B)$ to get the required result.

6 The Mechanics question

Since $ABCD$ is a square, the three bits of wire are identical. If B is at $(0, 0)$, A at $(2a, 0)$, C at $(0, 2a)$ and D at $(2a, 2a)$ then the centre of mass will be at:

$$\frac{(a,0) + (0,a) + (a,2a)}{3}.$$

As before, you need a neat diagram and it helps to draw in the line that passes through G and is perpendicular to AB.

STEP Probability/Statistics question

7 Preparation

(i) The vertical asymptote has equation $x = \frac{5}{2}$. If you divide throughout by x, you get $y = \dfrac{3}{\frac{5}{x} - 2}$ which means that $y \to -\frac{3}{2}$ as $x \to \infty$. The graph shows that y is increasing in the interval $-1 \leqslant x \leqslant 2$, so the minimum value of y occurs when $x = -1$ and the maximum value occurs when $x = 2$.

(ii) The asymptotes here are $x = \frac{5}{2}$ and $y = \frac{1}{2}$ and y is decreasing in the interval l $-1 \leqslant x \leqslant 2$, so the maximum occurs when $x = -1$ and the minimum occurs when $x = 2$.

(iii) **(a)** Since A and B are independent you should find that P(A|B) = P(A) i.e. whether B happens or not has no effect on the probability that A happens.

(b) The first thing you need to find is P(A ∩ B) = P(A) × P(B|A). You can then use the given results to find P(A|B) and P(A ∪ B).

(c) You should find that the probability that the Martian has tentacle rot, given that he or she has tested positive, is $\frac{8}{107}$ i.e. about 7.5%.

8 The Probability question

You will need the probability that a chip is sound, P(S) = 2p × 0.7 + p × 0.8 + (1 − 3p) × 0.9 along with P(A∩S) — the probability that a chip comes from A and is sound — and P(C∩S).

You can sketch both P(A|S) and P(C|S) and hence find the maximum values in the range $0 \leqslant p \leqslant \frac{1}{3}$.

Taking a sample of just one chip is not a very good way of estimating p.