STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 33

STEP I question

1 Preparation

(i) Here you could use partial fractions, various substitutions (including $y = x^2$, $y = 1 - x^2$ and $y = \sin\theta$) or use the standard result for $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\mathrm{d}x$.

(ii) Separate the variables, and it is probably easiest to find the constant of integration before you rearrange to get y in terms of x. There will be a vertical asymptote when $2\ln x = 1$ i.e. $x^2 = e$ and as $x \to +\infty$, $y \to 0$. The graph will only be defined for $x > 0$.

(iii) Since $y^2 = 3x(x^2 - 1)$, y will be real when $3x(x^2 - 1) \geqslant 0$. A sketch of $\mathrm{f}(x) = 3x(x^2 - 1)$ will help determine the required values of x.

2 The STEP I question

(i) Using the given substitution results in a differentiable equation in u in which the variables are separable. You can find the constant of integration before or after substituting $u = y/x$.

(ii) You can use the substitution $y = ux$ again, but the substitution $y = ux^2$ will result in a simpler differential equation. You will need to write down the values of x for which y is real, which requires the expression in the square root to be greater than or equal to zero.

(iii) Here you can use $y = ux^2$ to get a differential equation in which the variables are separable. You could instead do two substitutions, such as $y = ux$ followed by $u = vx$.

STEP III question

3 Preparation

(i) If $f'(x) \geqslant 0$ then the gradient is positive, and the graph is increasing. This means that $f(x) \geqslant 0$ for $x \geqslant 0$.

(ii) The first three parts and part **(e)** all need the chain rule. Part **(d)** needs the product rule.

(iii) The $Ax \sin x$ term can be integrated by parts.

(iv) If you set $y = f(x)$ you will end up with $\int y e^y \, dy$ which can be integrated by parts.

(v) You can differentiate $\cosh x$ and find the minimum value, showing that it is a minimum. A slightly more elegant way is to show that $\cosh x - 1 = \dfrac{e^x + e^{-x} - 2}{2} = \dfrac{(e^{x/2} - e^{-x/2})^2}{2}$ and hence $\cosh x - 1 \geqslant 0$.

4 The STEP III question

(i) Differentiating $\mathrm{E}(x)$ gives $\dfrac{\mathrm{dE}}{\mathrm{d}x} = 2\dfrac{\mathrm{d}y}{\mathrm{d}x}\left(\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} + y^3\right) = 0$ (using the given condition on y) and hence $\mathrm{E}(x)$ is constant. Using the initial conditions will give the value of the constant. You can then use $\frac{1}{2}y^4 = \mathrm{E} - \left(\dfrac{\mathrm{d}y}{\mathrm{d}x}\right)^2$ to show the required limits on y.

(ii) Differentiate $\mathrm{E}(x)$ and use $\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2} + \sinh v = -x\dfrac{\mathrm{d}v}{\mathrm{d}x}$ to show the first result. You can then use the initial conditions and the fact that $2\cosh v = \mathrm{E}(x) - \left(\dfrac{\mathrm{d}v}{\mathrm{d}x}\right)^2$ to deduce the second result.

(iii) The first two parts suggest that a good approach might be to consider a function $\mathrm{E}(x)$. By comparison with the first two parts the bit of $\mathrm{E}(x)$ which does not include a derivative is twice the integral of the bit in the differential equation which does not involve a derivative — i.e. in part **(i)** you consider $2\int y^3 \mathrm{d}y$.

By using this we get:

$$\mathrm{E}(x) = \left(\frac{\mathrm{d}w}{\mathrm{d}x}\right)^2 + 2\int (w\cosh w + 2\sinh w)\mathrm{d}w\,.$$

We can then show that $\dfrac{\mathrm{dE}}{\mathrm{d}x} \leqslant 0$, but this does involve showing that $5\cosh x - 4\sinh x - 3 \geqslant 0$ (which can be done by writing the hyperbolic functions in exponentials and using $(\mathrm{e}^2 - 3)^2 \geqslant 0$).

STEP Mechanics question

5 Preparation

No questions were asked here.

6 The Mechanics question

You first need to find the velocity vector, which will be of the form:

$$\dot{\mathbf{r}} = \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t \cos t\right)\right)\mathbf{i} + \left(\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathrm{e}^t \sin t\right)\right)\mathbf{j}\,.$$

Then you can use $\dot{\mathbf{r}} \cdot \mathbf{r} = |\dot{\mathbf{r}}||\mathbf{r}| \cos\theta$ to show that the angle between $\dot{\mathbf{r}}$ and $\mathbf{r}$ is $\theta = \frac{\pi}{4}$.

A similar argument can be used to show that the angle between $\ddot{\mathbf{r}}$ and $\mathbf{r}$ is $\frac{\pi}{2}$.

When sketching the path, actually plotting a few points might be helpful (such as when $t = 0, \frac{\pi}{4}, \frac{\pi}{2}$ etc.). The particle moves on a spiral moving further and further away from the origin.

Particle Q is T seconds behind P, so its displacement at time t will be:

$$\mathbf{r}_Q = \left(\mathrm{e}^{t-T} \cos(t-T)\right)\mathbf{i} + \left(\mathrm{e}^{t-T} \sin(t-T)\right)\mathbf{j}\,.$$

You can then consider $|\mathbf{r}_P - \mathbf{r}_Q|^2$ to show that the distance between P and Q is constant. Remember that T is a constant, so anything involving T will also be constant, such as $\cos T$, e^{-T} etc.

STEP Probability/Statistics question

7 Preparation

(i) **(a)** This is just the probability of HH, so $\frac{1}{4}$.

(b) Here we must have tossed TT followed by HH, so $\frac{1}{4} \times \frac{1}{4}$.

(c) The probability here is:

$$\frac{1}{4} + \left(\frac{1}{4}\right)^2 + \left(\frac{1}{4}\right)^3 + \cdots = \frac{\frac{1}{4}}{1 - \frac{1}{4}} = \frac{1}{3}\,.$$

The probabilities for Q and R are the same (by the same argument), so we have successfully used a fair coin (with 2 sides) to make a fair choice between 3 objects.

(ii) If the first toss is a Head then on the next toss we will either choose P or Q. If the first toss is a Tail then we will keep tossing until a Head comes up, when we will choose R (so the run TTTTTTTTH will result in choosing R). Hence we have $\mathrm{P}(P) = \frac{1}{4}$, $\mathrm{P}(Q) = \frac{1}{4}$ and $\mathrm{P}(R) = \frac{1}{2}$.

8 The Probability question

Careful explanation is needed through out. Just stating $\frac{1}{4}$ will not get you very far.

(i) If the first 2 tosses are "HH" then A will win a soon as the first T is tossed. For any of the other three possibilities (HT, TH and TT) B will win as the sequence "THH" must occur before "HHT".

(ii) Now all four players are playing, but as before if the first two tosses are "HH" A will win — we need a T for any of the others to win, but as soon as a T arrives A has won. If any of the other options for the first two tosses occur then A cannot win, so $\mathrm{P}(A) = \frac{1}{4}$.

By symmetry, the same argument can be used for C. Also by symmetry we can argue that $\mathrm{P}(B) = \mathrm{P}(D)$ and these must sum to $\frac{1}{2}$.

We hence have a "fair" game.

(iii) If the first two tosses are TT then C must win (as we will get 1 Head before 2 Heads).

The outcomes only depend on the previous two tosses. If we start with HT and then the next toss is a T then we are now in the same position as if the first two tosses were TT and the probability that C now wins is 1. If the next toss was a H then we are in the same position as if the first two tosses were TH and C will win with probability q.

This gives us the required $p = \frac{1}{2} \times 1 + \frac{1}{2} \times q$.

You then need to run through the other cases — so if the first two tosses were TH then after the next toss we will have either THH and B wins or THT which is the same as starting with HT and C will win with probability p. This gives $q = \frac{1}{2} \times 0 + \frac{1}{2} \times p$.

Once you have three equations you can solve for p, q and r. The probability that C will win is then $\frac{1}{4}(1 + p + q + r)$.

The odds are surprisingly skewed in C's favour!