STEP CORRESPONDENCE PROJECT

Postmortem: Assignment 7

Warm-up

1 **(i)** **(a)** For $y = x + \dfrac{1}{x}$, there are two turning points. For $y = x - \dfrac{1}{x}$, the gradient is positive for all (non-zero) x so there are no stationary points (the graph heads "upwards" at each point).

(b) Most people correctly stated that $y \to +\infty$ as $x \to +\infty$ and $y \to -\infty$ as $x \to -\infty$, but we can do a little better than that. As $x \to \pm\infty$, the $\pm\dfrac{1}{x}$ term becomes negligible, so we can say that $y \to x$ as $x \to \pm\infty$, in both cases.

To describe this behaviour, we say that $y = x$ is an *asymptote* — which means that the gap between the graph and the line $y = x$ approaches zero as $x \to \pm\infty$. It is good to draw the asymptotic line on the graph.

(c) The behaviour as $x \to 0$ depends on whether x is positive or negative. Consider the first graph:

$$y = x + \frac{1}{x}\,.$$

As x tends to 0 from the positive direction, y tends to $+\infty$, but as x tends to 0 from the negative direction, y tends to $-\infty$. We can write this very conveniently as:

$$\text{as } x \to 0_+,\ y \to +\infty \text{ and as } x \to 0_-,\ y \to -\infty\ .$$

(d) Neither graph intersects the y-axis (which would require $x = 0$). For the second graph $y = 0$ when $x = \pm 1$, so it intersects the x-axis at these values of x. For the first graph, there are no intersections with the x-axis since there are no (real) values of x that satisfy the equation $y = 0$.

(ii) **(a)** Multiplying throughout by x is a **bad** idea: when x is negative the inequality sign will be reversed, which makes it rather complicated. Instead draw the line $y = 2$ onto your graph of $y = x + \dfrac{1}{x}$.

Since the inequality is strict, you must be careful to exclude $x = 1$ (which gives equality) from your set of values.

(b) Replace the inequality sign with an equal sign and solve the resulting equation. Then sketch the line $y = \frac{3}{2}$ onto your graph to see where the equality holds.

Preparation

2 **(i)** When dividing by $(\alpha - \beta)$ it is good practice to say that you can do so because it is given that $\alpha \neq \beta$. If $\alpha \neq \beta$ were not given, then you would need to consider the case $(\alpha - \beta = 0)$ separately.

(v) There were 7 sets of (integer) values that work in the equation

$$(1+\alpha)(1+\beta)(1+\gamma) = -15$$

(though in our first draft we said there were 4 set of values, having forgotten the three involving 1, 1 and 15).

Now there are various ways to proceed. One way is to eliminate 6 of the 7 sets of values using the other equations. In fact, you only need one of the other equations in this case.

If you are taking this approach then you need to be systematic, and a table can help:

$1+\alpha$	$1+\beta$	$1+\gamma$	α	β	γ	$\alpha\beta\gamma$
1	1	−15	0	0	−16	0 ×
−1	−1	−15	−2	−2	−16	−64 ×
⋯	⋯	⋯	⋯	⋯	⋯	⋯
−1	3	5	−2	2	4	−16 ✓

You don't have to check the other equation because we are told that there is a solution of the given form.

The STEP question

3 To obtain the three given, results substitute in $x = 0$, $x = 1$ and $x = -1$. (Do be careful to write $(-1)^n$ and not -1^n.)

To find the roots, it is easiest to start with $(k_1 - 1)(k_2 - 1)(k_3 - 1)(k_4 - 1) = 175$. From this most people wrote down that $k_1, k_2, k_3, k_4 = 2, 6, 6, 8$, but there are other possibilities to be considered (as in Question 2) — 23 others, in fact. You can eliminate many by noting that $x = 0$ is not a root (so none of $k_1 - 1$, $k_2 - 1$ … can be -1).

You only have 3 equations with 4 unknowns, so just showing one set of values works in the 3 equations is not enough.

Probably the simplest attack is to expand $(x+2)(x+6)(x+6)(x+8)$ and show that this is the given quartic.

Please note that the question asks for the **roots** so the answer is **not** $2, 6, 6, 8$.

Warm down

4 This problem is known as "Bachet's Weights Problem", after Claude Gaspard Bachet de Méziriac (1581-1638) who published it in 1624, but it is thought to date back to Fibonacci in 1202, making it one of the earliest problems in integer partitions.

It was found hard!

(i) **(a)** If we are only putting weights on one side, then we obviously need a weight of 1oz, and we need either another 1oz or a 2oz to make 2oz. With the 2oz we can make 1oz, 2oz and 3oz (whereas is we had picked a second 1oz we would only be able to make 1oz or 2oz). If we then add a 4oz we can make 1oz, 2oz, 3oz, 4oz, 5oz, 6oz and 7oz etc.

(b) Each weight is either in the pan or not, so for each weight there are two options and with n weights there are $2 \times 2 \times \cdots \times 2 = 2^n$ options in total. Note that the weights have to be carefully picked if you are going to achieve 2^n *different* weights; for example, if you chose 1oz, 2oz, 3oz and 6oz, then two different combinations, $(1 + 2 + 3)$oz or 6oz, give the *same* weight.

There a connection with binary numbers here — the number 10 can be written as 1010, which is one 8, no 4's, one 2 and no 1's. The number 31 is 11111 — i.e. one 16, one 8, one 4, one 2 and one 1. The 1's and 0's correspond to whether a weight is in the pan or not.

If you have the weights $1, 2, 4, \cdots, 2^{n-1}$ then you can make all the numbers up to $1 + 2 + 4 + \cdots + 2^{n-1} = 2^n - 1$.

(ii) **(a)** 1oz and 3oz.

(b) There are **three** options for each weight this time: left pan, right pan or no pan. However 3^n is **not** achievable here.

To find the total number of distinct non-zero weights we can weigh, start by considering the 3^n possible distribution of our weights in the two pans. Take off 1 for the case where no weights are in either pan, which leaves us with $3^n - 1$ distributions. Then for each distribution, there is a "mirror image" where the weights swap pans. Both a given distribution and its mirror distribution weigh the same, so we need to divide by 2 giving at most $\dfrac{3^n - 1}{2}$ distinct weights.

(c) Guided by the previous parts, we try 1, 3, 9 and 27 oz, which works (and is in fact the only possibility with four weights).

Bachet proposed that the weights of $1, 3, 3^2, \cdots, 3^n - 1$ ounces enable one to weigh any number of weights from 1oz to $(1 + 3 + 3^2 + \cdots + 3^{n-1}) = \frac{1}{2}(3^n - 1)$ but he did not prove that this was the least possible number of weights. Major Percy MacMahon (1854-1929) proved this and that this solution is unique (so, for example, there is no other set of 4 weights that can weigh all integer weights from 1oz to 40oz).