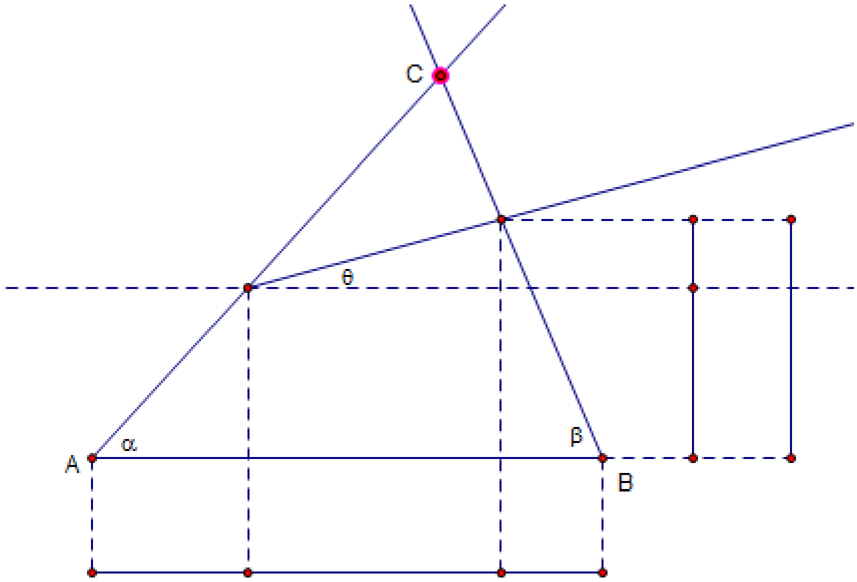
Question 1

(i)



The line of length 1 from A to B is divided into three sections with lengths $x\cos\alpha$, $x\cos\theta$ and $x\cos\beta$. Therefore $x\cos\alpha + x\cos\theta + x\cos\beta = 1$

So $x\cos\theta = 1 - x\cos\alpha - x\cos\beta$ **B1 (AG)**

Considering the vertical distances:

$$x\sin\alpha + x\sin\theta = x\sin\beta$$

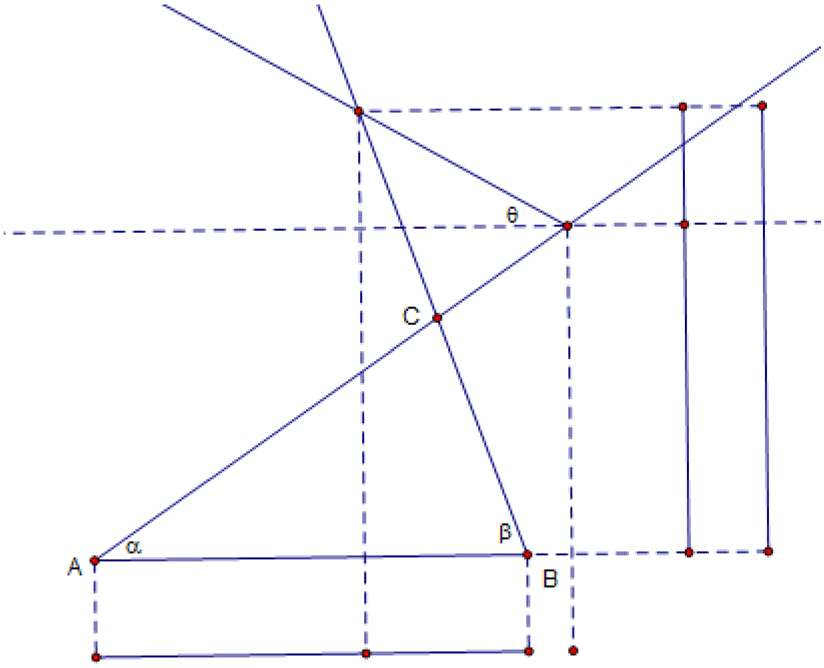
Rearranging gives $x\sin\theta = x\sin\beta - x\sin\alpha$ **M1 A1**

Squaring and adding:

$$(1 - x\cos\alpha - x\cos\beta)^2 + (x\sin\beta - x\sin\alpha)^2 = x^2$$ **M1**

$$(1 - 2\sin\alpha\sin\beta + 2\cos\alpha\cos\beta)x^2 - 2(\cos\alpha + \cos\beta)x + 1 = 0$$

$$(1 + 2\cos(\alpha+\beta))x^2 - 2(\cos\alpha + \cos\beta)x + 1 = 0$$ **M1 A1 (AG)**



B1

B1

B1

B1

Horizontally the relationship is $x\cos\alpha - x\cos\theta + x\cos\beta = 1$ and the vertical relationship is unchanged, so when squaring and adding the same result will follow

(ii) To be linear the coefficient of x^2 must be 0. Therefore $\cos(\alpha+\beta) = -\frac{1}{2}$

$$\alpha + \beta = \frac{2\pi}{3} \; or \; 120°$$ **B1**

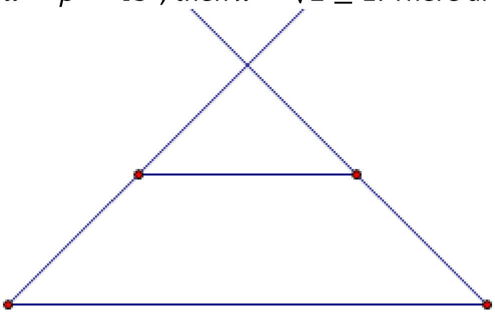
If it is not linear then the discriminant is

$$\big(2(\cos\alpha + \cos\beta)\big)^2 - 4(1 + 2\cos(\alpha+\beta))(1)$$

which simplifies to $8\sin\alpha\sin\beta$.

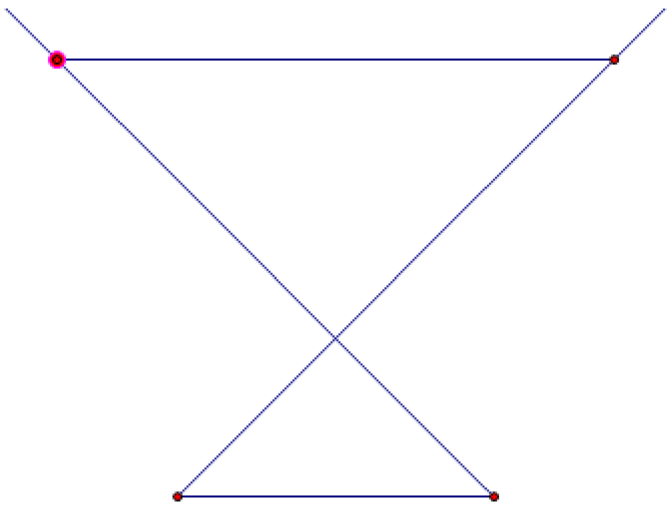
Since both $\sin\alpha > 0$ and $\sin\beta > 0$ for values in the range stated, the discriminant must be positive and so there are distinct real roots. **M1** **M1 A1**

(iii) (a) If $\alpha = \beta = 45°$, then $x = \sqrt{2} \pm 1$. There are two diagrams **B1**



$(\sqrt{2} - 1)$

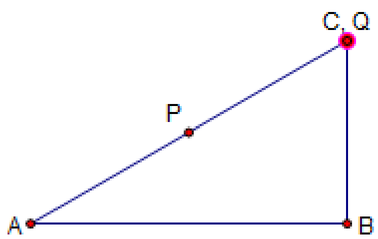
B1



$(\sqrt{2} + 1)$ **B1**

(b) If $\alpha = 30°, \beta = 90°$, then $x = \frac{\sqrt{3}}{3}$ **B1**

Diagram:



B1
B1

(i)	
B1	For a justification of the given relationship (lengths marked on a diagram is sufficient)
M1	For consideration of vertical distances to points P and Q
A1	For correct expression for $x\sin\theta$
M1	For squaring and adding the two expressions
M1	For reaching a point where the substitution for $\cos(\alpha+\beta)$ can be made
A1	For reaching the given solution
B1	A suitable diagram to represent the situation (award this mark if a convincing argument is made without a diagram)
B1	Identifying that the expression for $x\cos\theta$ is multiplied by -1
B1	For identifying that the expression for $x\sin\theta$ is unchanged
B1	For observing that there will be no difference to the equation once the expressions are squared and added
(ii)	
B1	For obtaining the condition on α and β
M1	For an attempt to calculate the discriminant
M1	For reaching a point at which it can be observed that the discriminant is always positive
A1	For concluding that there are distinct real roots
(iii) (a)	
B1	For obtaining two values of x
B1	For a correct diagram where the line PQ is parallel to AB in both cases
B1	For two correct diagrams drawn
(iii) (b)	
B1	For obtaining the correct value of x
B1	For a diagram showing P as the midpoint of AC
B1	For a diagram showing Q at the same point as C

Question 2

(i)

$$\int_0^\pi \sin^2 nx\,dx = \int_0^\pi \frac{1}{2}(1-\cos 2nx)dx$$
$$= \left[\frac{1}{2}x - \frac{1}{4n}\sin 2nx\right]_0^\pi$$
$$= \frac{\pi}{2}$$ **M1 A1**

$$f'(x) = n\cos nx$$

$$\int_0^\pi n^2\cos^2 nx\,dx = \int_0^\pi \frac{n^2}{2}(1+\cos 2nx)dx$$
$$= \left[\frac{n^2}{2}x + \frac{n}{4}\sin 2nx\right]_0^\pi$$
$$= \frac{n^2\pi}{2}$$ **A1**

Since n is a positive integer, (*) must be satisfied. **B1**

For example, if n did not have to be an integer in the calculations above, the integrals would give values of $\frac{\pi}{2} - \frac{1}{4n}\sin 2n\pi$ and $\frac{n^2\pi}{2} + \frac{n}{4}\sin 2n\pi$.

A value of n is required such that $\frac{n^2\pi}{2} + \frac{n}{4}\sin 2n\pi < \frac{\pi}{2} - \frac{1}{4n}\sin 2n\pi$

Assuming n is positive:

$$2n^3\pi + n^2\sin 2n\pi < 2\pi - \sin 2n\pi$$
$$\sin 2n\pi < \frac{2\pi(1-n^3)}{n^2+1}$$

A small enough value of n to guarantee that $\frac{2\pi(1-n^3)}{n^2+1} > 1$ would be sufficient. **M1 A1**

Having found a counterexample where $f(0) = 0, f(\pi) \neq 0$, the function $g(x) = f(\pi - x)$ will provide a counterexample where $g(\pi) = 0, g(0) \neq 0$ **M1 A1**

(ii) Since it is required that $f(0) = 0, f(\pi) = 0$, the function should be

$$f(x) = x^2 - \pi x$$ **B1**

Left hand side of (*):

$$\int_0^\pi x^4 - 2\pi x^3 + \pi^2 x^2\,dx = \left[\frac{1}{5}x^5 - \frac{\pi}{2}x^4 + \frac{\pi^2}{3}x^3\right]_0^\pi = \frac{\pi^5}{30}$$ **B1**

$f'(x) = 2x - \pi$:

Right hand side of (*):

$$\int_0^\pi 4x^2 - 4\pi x + \pi^2\,dx = \left[\frac{4x^3}{3} - 2\pi x^2 + \pi^2 x\right]_0^\pi = \frac{\pi^3}{3}$$ **B1**

Therefore, substituting into (*) gives $\pi^2 \leq 10$ **B1**

The conditions that $f(0) = 0, f(\pi) = 0$ lead to the following equations:

$$q + r = 0$$
$$p + r = 0$$

Therefore let $p = q = 2, r = -2$ **M1 A1**

Left hand side of (*):

$$4\int_0^{\pi} 2 - 2\sin\tfrac{1}{2}x - 2\cos\tfrac{1}{2}x + 2\sin\tfrac{1}{2}x\cos\tfrac{1}{2}x\, dx$$
$$= 4\int_0^{\pi} 2 - 2\sin\tfrac{1}{2}x - 2\cos\tfrac{1}{2}x + \sin x\, dx$$
$$= 4\left[2x + 4\cos\tfrac{1}{2}x - 4\sin\tfrac{1}{2}x - \cos x\right]_0^{\pi} = 4(2\pi - 6)$$ **M1 A1**

$f'(x) = \cos\tfrac{1}{2}x - \sin\tfrac{1}{2}x$:

Right hand side of (*):

$$\int_0^{\pi} 1 - 2\cos\tfrac{1}{2}x\sin\tfrac{1}{2}x\, dx = \int_0^{\pi} 1 - \sin x\, dx$$
$$= [x + \cos x]_0^{\pi} = \pi - 2$$ **M1 A1**

Therefore, substituting into (*) gives $8\pi - 24 \leq \pi - 2$, so $\pi \leq \frac{22}{7}$. **B1**

$\left(\frac{22}{7}\right)^2 < 10$, so $\pi \leq \frac{22}{7}$ leads to a better estimate for π^2. **B1**

(i)	
M1	For using substitution of $\cos 2nx$
A1	For correct integration in terms of x and evaluation of limits
A1	For correct integration in terms of x and evaluation of limits
B1	Conclusion that (*) is satisfied
M1	For attempting to verify that a function with $f(0) = 0, f(\pi) \neq 0$ is a counterexample to (*) *Some attempt to evaluate integrals must be made to receive this mark*
A1	For showing that the inequality is not satisfied
M1	For attempting to verify that a function with $f(0) \neq 0, f(\pi) = 0$ is a counterexample to (*) *Some attempt to evaluate integrals must be made to receive this mark*
A1	For showing that the inequality is not satisfied
(ii)	
B1	Identification of the function $f(x)$ (multiples of the given function are also possible)
B1	For evaluation of the integral on the left of (*) as a function of x
B1	For evaluation of the integral on the left of (*) as a function of x
B1	For establishing the given inequality using (*)
M1	For establishing the simultaneous equations
A1	For choosing values for p, q and r to fit the equations
M1	For integration of $(f(x))^2$
A1	For evaluation of the limits
M1	For integration of $(f'(x))^2$
A1	For evaluation of the limits
B1	For establishing the inequality
B1	For justifying which gives the better estimate

Question 3

(i) The straight line will meet the x-axis at the point $\left(\frac{c}{m}, 0\right)$ and the y-axis at the point $(0, c)$. Let the closest distance to the origin be d. **B1**

The length of the line joining these two points is $\frac{c}{m}\sqrt{m^2+1}$. **B1**

Using similar triangles we have that:

$$\frac{c/m}{c\sqrt{m^2+1}/m} = \frac{d}{c}$$

$$d = c/\sqrt{m^2+1}$$ **M1 A1**

(ii) The gradient of the tangent is y'. Therefore the point of intersection of the tangent with the y-axis will be $y - xy'$ **M1 A1**

Therefore (using the result from (i)):

$$a = y - xy' \Big/ \sqrt{(y')^2+1}$$

Which simplifies to give **M1 A1**

$$(y - xy')^2 = a^2(1 + (y')^2)$$ **(AG)**

Differentiating with respect to x:

$$2(y - xy')(y - xy'' - y') = 2a^2y'y''$$ **M1 A1**

$$y''\left(a^2y' + x(y - xy')\right) = 0$$ **B1**

Therefore, either $y'' = 0$ or $a^2y' + x(y - xy') = 0$

If $y'' = 0$:

The general solution will be $y = mx + c$ and from part (i) it follows that $c = a\sqrt{1+m^2}$ if $c \geq 0$. By symmetry, the solution with $c \leq 0$ must be $c = -a\sqrt{1+m^2}$.

Therefore the equation of the line is $y = mx \pm a\sqrt{1+m^2}$ **M1 A1**

If $a^2y' + x(y - xy') = 0$:

$$(a^2 - x^2)y' = -xy$$

Therefore:

$$\int \frac{1}{y}dy = \int \frac{-x}{a^2 - x^2}dx$$

Integrating:

$$\ln ky = \frac{1}{2}\ln|a^2 - x^2|$$ **M1 A1**

Therefore:

$$k^2y^2 = a^2 - x^2$$

$k = 1$, so $x^2 + y^2 = 1$ **M1 A1**

(iii)

Any arc of a circle centred on the the origin and radius a, with straight lines extending from the end points of the arc at each side. **B1** **B1** **B1**

(i)	
B1	For a diagram showing the line and the closest distance to the origin
B1	For calculation of the length of the hypotenuse of the right angled triangle
M1	For use of similar triangles
A1	For establishing the required value for d
(ii)	
M1	For identifying the gradient of the tangent
A1	For finding the point of intersection with the y axis
M1	For substituting into the result from part (i)
A1	For reaching the given equation
M1	For differentiating the equation (either side differentiated correctly
A1	For differentiation of both sides correct
B1	For rearranging and factorising
M1	For solving the differential equation y’’=0
A1	For identifying the two possible values for c (the negative value must be justified)
M1	For separating the variables
A1	For correctly integrating the expressions
M1	For removing the logs from the expression
A1	For reaching the required equation
(iii)	
B1	Arc of circle centred on the origin and radius a
B1	No vertical tangents to the arc of the circle
B1	Straight lines extending from each end of the arc.

Question 4

(i)

$$x = b \quad u = \tfrac{1}{b}$$

$$x = \tfrac{1}{b} \quad u = b$$ **B1**

$$\frac{dx}{du} = -\frac{1}{u^2}$$ **B1**

$$I = \int_{1/b}^{b} \frac{x \ln x}{(a^2 + x^2)(a^2x^2 + 1)} dx$$

becomes

$$\int_{b}^{1/b} \frac{\frac{1}{u}\ln\frac{1}{u}}{\left(a^2 + \left(\frac{1}{u}\right)^2\right)\left(\frac{a^2}{u^2} + 1\right)} \left(-\frac{1}{u^2}\right) du$$

which simplifies to

$$-\int_{1/b}^{b} \frac{u \ln u}{(a^2 + u^2)(a^2u^2 + 1)} du$$ **M1 M1**

M1

Therefore $I = -I$ and so $I = 0$. **A1 (AG)**

(ii)

$$I = \int_{1/b}^{b} \frac{\arctan x}{x} dx$$

becomes

$$\int_{b}^{1/b} \frac{\arctan\frac{1}{u}}{\frac{1}{u}} \left(-\frac{1}{u^2}\right) du$$

$$\arctan\tfrac{1}{u} = \frac{\pi}{2} - \arctan u$$

So I simplifies to

$$\int_{1/b}^{b} \frac{\pi}{2x} - \frac{\arctan x}{x} dx$$ **M1 A1**

M1 A1

Therefore

$$2I = \int_{1/b}^{b} \frac{\pi}{2x} dx$$

A1

Therefore $I = \frac{\pi \ln b}{2}$ **M1 A1**

(AG)

(iii) Let $u = \frac{k}{x}$ for some constant k:
Then the integral becomes

$$I = \int_{\infty}^{0} \frac{1}{\left(a^2 + \frac{k^2}{u^2}\right)^2} \left(-\frac{k}{u^2}\right) du$$

which simplifies to

$$\int_{0}^{\infty} \frac{ku^2}{(a^2u^2 + k^2)^2} du$$

M1 A1

So we will need to use $k = a^2$ **B1**

The integral therefore becomes

$$I = \frac{1}{a^2} \int_{0}^{\infty} \frac{u^2}{(a^2 + u^2)^2} du$$

Splitting the integral gives

$$I = \frac{1}{a^2} \int_{0}^{\infty} \frac{1}{a^2 + u^2} du - \frac{1}{a^2} \int_{0}^{\infty} \frac{a^2}{(a^2 + u^2)^2} du$$ **M1 A1**

Using the given result:

$$I = \frac{\pi}{2a^3} - I$$

which leads to

$$I = \frac{\pi}{4a^3}$$ **M1 A1 (AG)**

(i)	
B1	For calculation of limits following substitution
B1	For calculation of dx/du
M1	For correct substitution throughout integral into terms of u
M1	Multiplication of top and bottom by u^4
M1	Use of $\ln \frac{1}{u} = -\ln u$
A1	Explanation of how this leads to the value of the integral being 0
(ii)	
M1	For correct substitution throughout integral into terms of u
A1	For simplification of expression
M1	For use of $\arctan \frac{1}{u} = \frac{\pi}{2} - \arctan u$
A1	For simplification of expression
A1	For finding an expression for I that can be evaluated
M1	For integrating the function
A1	For reaching the given answer
(iii)	
M1	For substitution into the given integral
A1	For simplifying to an expression from which the appropriate value of k can be deduced
B1	For selection of value of k
M1	For attempting to split the integral into two
A1	For a correct split
M1	For using the given result
A1	For a convincing solution to the given answer

Question 5

(i)

$$\left(\frac{dy}{dx}=\right)x\frac{du}{dx}+u \qquad \textbf{B1}$$

$$x\frac{du}{dx}+u=\frac{2xu+x}{xu-2x}$$

$$x\frac{du}{dx}+u=\frac{2u+1}{u-2} \qquad \textbf{M1 A1}$$

Therefore

$$\frac{du}{dx}=\frac{1+4u-u^2}{u-2} \qquad \textbf{M1 A1}$$

$$\int\frac{u-2}{1+4u-u^2}\,du=\int\frac{1}{x}\,dx$$

$$-\frac{1}{2}\ln|1+4u-u^2|=\ln|kx| \qquad \textbf{M1 A1}$$

Since the curve passes through (1,1):

$$-\frac{1}{2}\ln 4=\ln k$$

Therefore $k=\frac{1}{2}$ **M1 A1**

Therefore

$$1+4u-u^2=\frac{4}{x^2}$$

and so

$$x^2+4xy-y^2=4 \qquad \textbf{M1 A1}$$

(ii)

$$\frac{dY}{dX}=\frac{dY}{dx}\times\frac{dx}{dX}$$

Since $\frac{dY}{dx}=\frac{dy}{dx}$ and $\frac{dX}{dx}=1$, we have that $\frac{dY}{dX}=\frac{dy}{dx}$ **M1 A1**

$$\frac{dY}{dX}=\frac{X+a-2(Y+b)-4}{2(X+a)+(Y+b)-3}$$

To have an expression similar to that in part (i) we need:

$$a-2b-4=0$$

and

$$2a+b-3=0 \qquad \textbf{B1}$$

Therefore, $a=2, b=-1$. **M1 A1**

This gives

$$\frac{dY}{dX}=\frac{X-2Y}{2X+Y}$$

or

$$\frac{dX}{dY}=\frac{2X+Y}{X-2Y} \qquad \textbf{B1}$$

The solution must be of the form

$$Y^2+4XY-X^2=c \qquad \textbf{B1}$$

Therefore

$$(y-1)^2+4(y-1)(x+2)-(x+2)^2=c$$

Substituting in (1,1), gives $c=9$

$$(y-1)^2+4(y-1)(x+2)-(x+2)^2=9 \qquad \textbf{M1 A1}$$

B1	For the correct expression
(i)	
M1	For substitution into the differential equation
A1	For cancelling the factor of x
M1	For grouping all terms of u together
A1	For reaching the correct quadratic in the numerator
M1	For separating the variables and integrating both sides
A1	For correct integrations of both sides
M1	For substitution of point on curve
A1	For finding value of constant of integration
M1	For substituting back in terms of x and y
A1	For reaching the equation of the curve
(ii)	
M1	For attempting to justify that $\frac{dY}{dX} = \frac{dy}{dx}$
A1	For full justification of $\frac{dY}{dX} = \frac{dy}{dx}$
B1	For establishing the simultaneous equations
M1	For attempting to solve the simultaneous equations
A1	For the correct values of a and b
B1	For rearranging to have a differential equation similar to that in (i)
B1	For stating the solution to the equation in terms of X and Y
M1	For substituting back to x and y and substituting in the point on the curve
A1	For the correct final equation

Question 6

$$\sin\left(r+\tfrac{1}{2}\right)x - \sin\left(r-\tfrac{1}{2}\right)x = 2\cos rx \sin\tfrac{1}{2}x$$ **M1 A1**

$$2\sin\tfrac{1}{2}x\,(\cos x + \cos 2x + \cdots + \cos nx) \quad = \quad \begin{array}{rcl} \sin\tfrac{3}{2}x & - & \sin\tfrac{1}{2}x \\ \sin\tfrac{5}{2}x & - & \sin\tfrac{3}{2}x \\ & \vdots & \\ \sin\tfrac{(n+1)}{2}x & - & \sin\tfrac{(n-1)}{2}x \end{array}$$

Therefore

$$\cos x + \cos 2x + \cdots + \cos nx = \frac{\sin\tfrac{(n+1)}{2}x - \sin\tfrac{1}{2}x}{2\sin\tfrac{1}{2}x}$$ **M1 A1 (AG)**

(i)

$$S_2(x) = \sin x + \tfrac{1}{2}\sin 2x$$

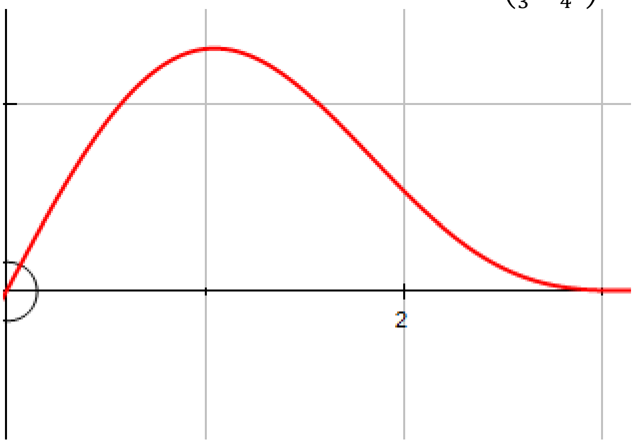
Therefore

$$S_2{}'(x) = \cos x + \cos 2x$$ **B1**

$$S_2'(x) = 2\cos^2 x + \cos x - 1$$ **M1**

So the stationary points occur when $\cos x = \frac{1}{2}$ or $\cos x = -1$

Stationary points in the given interval are $\left(\frac{\pi}{3}, \frac{3\sqrt{3}}{4}\right)$ and $(\pi, 0)$ **M1 A1**



B1

(ii)

$$S_n'(x) = \cos x + \cos 2x + \cdots + \cos nx$$ **B1**

Therefore, by the result from the start of the question:

$$S_n'(x) = \frac{\sin\tfrac{(n+1)}{2}x - \sin\tfrac{1}{2}x}{2\sin\tfrac{1}{2}x}$$ **B1**

At stationary points:

$$\frac{\sin\tfrac{(n+1)}{2}x - \sin\tfrac{1}{2}x}{2\sin\tfrac{1}{2}x} = 0$$

Since $\sin\frac{1}{2}x \neq 0$ in the given range:

$$\sin\tfrac{(n+1)}{2}x = \sin\tfrac{1}{2}x$$

Therefore:

$$\sin nx_0 \cos\tfrac{1}{2}x_0 + \cos nx_0 \sin\tfrac{1}{2}x_0 = \sin\tfrac{1}{2}x_0$$ **M1 A1**

$0 < x_0 < \pi$, so $\sin\frac{1}{2}x_0 \neq 0$ and $\cos\frac{1}{2}x_0 \neq 0$: **B1**

Therefore

$$\sin nx_0 = (1 - \cos nx_0)\tan\tfrac{1}{2}x_0$$ **B1 (AG)**

$$S_n(x) = S_{n-1}(x) + \frac{1}{n}\sin nx$$ **B1**

Since $0 < x_0 < \pi$, $\sin nx_0 = (1 - \cos nx_0)\tan\frac{1}{2}x_0 \geq 0$

Therefore $S_n(x) \geq S_{n-1}(x)$ **B1**

At any stationary point of $S_n(x)$, $S_n(x) \geq S_{n-1}(x) > 0$.

Since $S_n(0) = S_n(\pi)$ and all stationary points have $S_n(x) > 0$, it must be true that $S_n(x) > 0$ for all values of x. **B1**

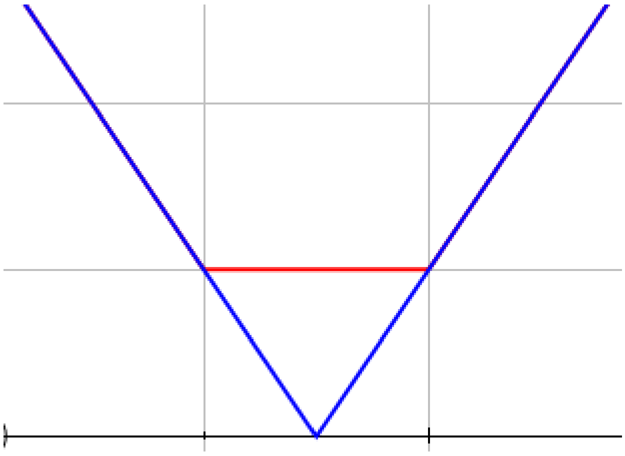
(iii) The result is true for S_1, and in part (ii) we have shown that if the result is true in the case $n = k$ then it will be true in the case $n = k + 1$.

Therefore, by induction, $S_n(x) \geq 0$ for $n \geq 1$ and $0 \leq x \leq \pi$ **M1 A1**

M1	For applying the formula for $\sin(A+B)$
A1	For simplifying the expression
M1	For expressing the sum using the previous result
A1	For deduction of the given result
(i)	
B1	For differentiating the function
M1	For expressing cos 2x in terms of cos x
M1	For solving the resulting equation for cos x
A1	For finding the correct stationary points of the curve
B1	For a sketch showing the two stationary points correctly (and no other stationary points)
(ii)	
B1	For writing down the correct derivative of the function
B1	For applying the result from the start of the question
M1	For applying the formula for $\sin(A+B)$
A1	For reaching the expression in terms of sin and cos
B1	For commenting that sin and cos are non-zero (so division is fine)
B1	For reaching the given result
B1	For splitting S_n
B1	For clear explanation of the result
B1	For justification of the result
(iii)	
M1	For an attempt to demonstrate the case n=1
A1	For an explanation of the inductive proof

Question 7

(i)



B1
B1
B1

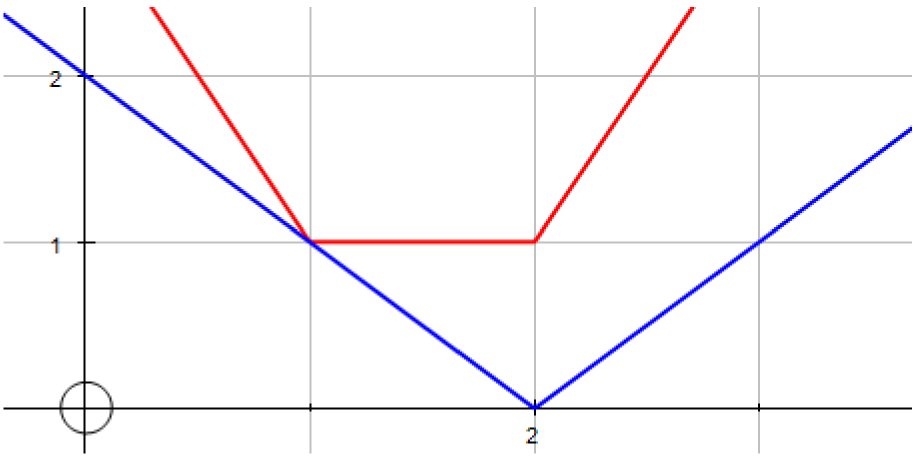
$(y = f(x)$ has two sloping sections and the horizontal section in the middle, $y =$ **B1**
$g(x)$ has two sloping sections and touches the x- axis **B1**

The gradients for $f(x)$ are:

$x < a$	-2
$a < x < b$	0 **B1**
$x > b$	2 **B1**

The quadrilateral is a square **B1**

(ii)



The diagram is the case where $c = b$. If $c > b$ then there will be two roots. If $c <$ **B1**
b then there will be no roots until the case where $c = a$. By symmetry we also **B1**
have two roots if $c < a$ **B1**
If $c < a$ or $c > b$ there are two roots **B1**
If $c = a$ or $c = b$ there is one root **B1**
If $a < c < b$ there are no roots **B1**

(iii) From the result in part (i), the roots can be determined by considering the graphs **B1**
of $y = |2x - a - b|$ and $y = |2x - c - d|$. **B1**
Since one graph is a translation of the other, there are either infinitely many **B1**
solutions, or just 1. **B1**
If $a + b = c + d$ there are infinitely many roots **B1**
If $a + b \neq c + d$ there is only one root **B1**

(i)	
B1	Graph of $y = f(x)$ has two sloping sections joined by a horizontal section
B1	Graph of $y = g(x)$ has two sloping sections
B1	Point of graph of $y = g(x)$ touches the x-axis
B1	Sloping sides of $y = g(x)$ are parallel to those of $y = f(x)$
B1	Sloping side of $y = f(x)$ overlap sloping lines of $y = g(x)$
B1	Gradients of 2 and -2 identified
B1	Gradient of 0 identified
B1	Identification of the quadrilateral as a square
(ii)	
B1	Diagram for either the case $c = a$ or $c = b$
B1	Identification that there will be one root in this case
B1	Identification of the second case where there is only one root
B1	Justification that there are two roots in the case $c < a$
B1	Justification that there are two roots in the case $c < b$
B1	Justification that there are no roots in the remaining cases
(iii)	
B1	Sketch of an example case (award this mark if there is a clear explanation for all other marks even if no sketch is drawn)
B1	Consideration of the graphs of $y = \|2x - a - b\|$ and $y = \|2x - c - d\|$
B1	Identification that one graph is a translation of the other
B1	Explanation that this means either 1 or infinitely many roots
B1	Identification of the condition for infinitely many roots
B1	Explanation that all other cases lead to just one root

Question 8

$$c_r = \binom{n}{r} a^{n-r} b^r = \frac{n!}{r!\,(n-r)!} a^{n-r} b^r$$ **B1**

$$\frac{c_{r+1}}{c_r} = \frac{b(n-r)}{a(r+1)}$$ **M1 A1**

$\frac{c_{r+1}}{c_r} > 1$if $b(n-r) > a(r+1)$

which simplifies to $r < \frac{nb-a}{a+b}$. **M1 A1**

Similarly, $\frac{c_{r+1}}{c_r} = 1$ if $r = \frac{nb-a}{a+b}$ and $\frac{c_{r+1}}{c_r} < 1$ if $r > \frac{nb-a}{a+b}$ **B1**

Therefore the maximum value of c_r will be the first integer after $\frac{nb-a}{a+b}$ (and there will be two maximum values for c_r if $\frac{nb-a}{a+b}$ is an integer.

Therefore:

$$\frac{nb-a}{a+b} \le m \le \frac{nb-a}{a+b} + 1$$

$$\frac{b(n+1)}{a+b} - 1 \le m \le \frac{b(n+1)}{a+b}$$ **B1**

This range either has two consecutive integers as its endpoints (in which case m is one of two consecutive integers) or only contains one integer value (in which case m is a unique integer). **B1**

(i) G(9,1,3):

$$\frac{3(9+1)}{1+3} - 1 \le m \le \frac{3(9+1)}{1+3}$$

$$6.5 \le m \le 7.5$$

Therefore $G(9,1,3) = 7$ **M1 A1**

G(9,2,3):

$$\frac{3(9+1)}{2+3} - 1 \le m \le \frac{3(9+1)}{2+3}$$

$$5 \le m \le 6$$

Therefore $G(9,2,3) = 6$ **M1 A1**

(ii) G(2k,a,a):

$$\frac{a(2k+1)}{2a} - 1 \le m \le \frac{a(2k+1)}{2a}$$

$$\frac{2k-1}{2} \le m \le \frac{2k+1}{2}$$

Therefore $G(2k, a, a) = k$ **M1 A1**

G(2k-1,a,a):

$$\frac{a(2k)}{2a} - 1 \le m \le \frac{a(2k)}{2a}$$

$$k - 1 \le m \le k$$

Therefore $G(2k-1, a, a) = k$ **M1 A1**

(iii) The greatest value will occur when $a + b$ is minimised. Therefore $G(n, a, b)$ is greatest when $a = 1$. **B1**

(iv) G(n,1,b):

$$\frac{b(n+1)}{1+b} - 1 \leq m \leq \frac{b(n+1)}{1+b}$$

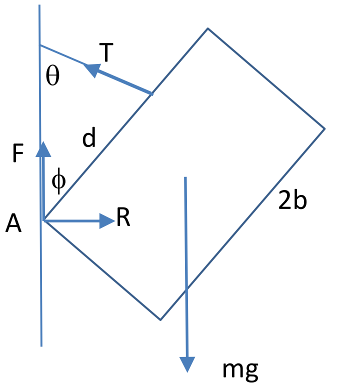
The value of $G(n,1,b)$ cannot be greater than n. **B1**

For $G(n,1,b) = n$, we need $\frac{b(n+1)}{1+b} \geq n$.

$$b(n+1) \geq n(1+b)$$
$$b \geq n$$ **M1 A1**

B1	For a correct expression for c_r
M1	For attempting to calculate $\frac{c_{r+1}}{c_r}$
A1	For a correct expression for $\frac{c_{r+1}}{c_r}$
M1	For considering when $\frac{c_{r+1}}{c_r}$ is either greater than, equal to or less than 1
A1	For establishing a relevant inequality
B1	For a statement of how the value of $\frac{c_{r+1}}{c_r}$ changes as *r* changes
B1	For a justification of the inequality
B1	For deduction that m is either a unique integer or one of two consecutive integers (do not require a justification here that both values for c_r are equal when the inequality gives two values for m, but note that the two possible values will need to be checked for the final A mark in parts (i) and (ii) if not)
(i)	
M1	For substitution into the formula
A1	For choosing the integer value
M1	For substituting into the formula
A1	For choosing the higher of the two integer values (if there was no clear statement that both values for c_r are equal when the inequality gives two values for m, then the two values must be checked for this mark)
(ii)	
M1	For substitution into the formula
A1	For choosing the integer value
M1	For substituting into the formula
A1	For choosing the higher of the two integer values (if there was no clear statement that both values for c_r are equal when the inequality gives two values for m, then the two values must be checked for this mark)
(iii)	
B1	For justification of the correct choice for a
(iv)	
B1	For identifying that n is the maximum value possible
M1	For obtaining the inequality that b must satisfy
A1	For finding the range of values for b (do not penalise candidates for not checking the two possible values for m in the boundary case)

Question 9



(i) Resolving vertically:

$$F + T\cos\theta = mg$$ **M1 A1**

Resolving horizontally:

$$T\sin\theta = R$$ **A1**

Taking moments about A: **B1**

$$mg(a\cos\varphi + b\sin\varphi) = Td\sin(\theta+\varphi)$$ **B1**

M1 A1

Limiting equilibrium, so $F = \mu R$: **B1**

$$\mu T\sin\theta + T\cos\theta = mg$$

Therefore:

$$Td\sin(\theta+\varphi) = T(\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

And so: **M1 A1**

$$d\sin(\theta+\varphi) = (\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$ **(AG)**

(ii) If the frictional force were acting downwards then the F in the first of the equations would be changed to –F and so the equation would become:

$$d\sin(\theta+\varphi) = (-\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$ **M1 A1**

(iii) In the case where friction is not limiting the first and third equations give:

$$(F + T\cos\theta)(a\cos\varphi + b\sin\varphi) = Td\sin(\theta+\varphi)$$

With F positive if it is directed upwards and negative if it is directed downwards **M1 A1**

Rearranging:

$$F = \frac{Td\sin(\theta+\varphi)}{a\cos\varphi + b\sin\varphi} - T\cos\theta$$

$$\frac{Td\sin(\theta+\varphi)}{a\cos\varphi + b\sin\varphi} - T\cos\theta > 0$$

So

$$d > \frac{\cos\theta\,(a\cos\varphi + b\sin\varphi)}{\sin(\theta+\varphi)}$$ **M1 A1**

Dividing top and bottom by $\cos\theta\cos\varphi$ gives:

$$d > \frac{a + b\tan\varphi}{\tan\theta + \tan\varphi}$$ **B1**

This cannot be satisfied if

$$\frac{a + b\tan\varphi}{\tan\theta + \tan\varphi} > 2b$$ **B1**

Which simplifies to $b(2\tan\theta + \tan\varphi) < a$ **B1 (AG)**

(i)	
M1	For resolving forces (either horizontally or vertically)
A1	For the correct equation from resolving horizontally
A1	For the correct equation from resolving vertically
B1	For obtaining the perpendicular distance from A to the line of action of the weight ($a\cos\varphi + b\sin\varphi$)
B1	For obtaining the perpendicular distance from A to the line of action of T ($d\sin(\theta+\varphi)$)
M1	For taking moments about A
A1	For obtaining the correct equation
B1	For statement that $F = \mu R$
M1	For eliminating R from the equations
A1	For a solution reaching the given equation
(ii)	
M1	For identifying that the only change would be the sign of F in the original three equations
A1	For clearly indicating how the final equation would be changed (it is not necessary to have stated the new equation)
(iii)	
M1	For attempting to eliminate mg from the original equations in the non-limiting case
A1	For reaching the correct formula
M1	For making F the subject of the formula and stating an inequality
A1	For making d the subject
B1	For dividing through by $\cos\theta\cos\varphi$
B1	For identifying that the length of the side of the rectangle restricts the possible values of d
B1	For reaching the given inequality

Question 10

Horizontally:

$$x = ut\cos\theta$$

So

$$t = \frac{x}{u\cos\theta}$$ **B1**

Vertically:

$$y = ut\sin\theta - \frac{1}{2}gt^2$$ **M1 A1**

Eliminating t:

$$y = x\tan\theta - \frac{gx^2}{2u^2\cos^2\theta}$$ **M1 A1**

Since $\lambda = \tan\theta$, $\sec^2\theta = 1 + \lambda^2$:

$$y = \lambda x - \frac{gx^2}{2u^2}(1 + \lambda^2)$$ **M1 A1**

Completing the square:

$$y = -\frac{gx^2}{2u^2}\left(\lambda - \frac{u^2}{gx}\right) + \frac{u^2}{2g} - \frac{gx^2}{2u^2}$$ **M1 A1**

Therefore the maximum value of y occurs when $\lambda = \frac{u^2}{gx}$ and is equal to:

$$\frac{u^2}{2g} - \frac{gx^2}{2u^2}$$ **M1 A1**

And as it is a quadratic function all values up to this maximum value are possible **B1**

Graph of $Y = \frac{u^2}{2g} - \frac{gx^2}{2u^2}$. **B1**

B1

The region in the first quadrant bounded by the curve and the axes is the set of points that can be reached by a particle projected from O with speed u. **B1**

B1

The greatest distance must lie on the curve. Let d be the distance from O to the point (x, Y):

$$d^2 = x^2 + Y^2$$

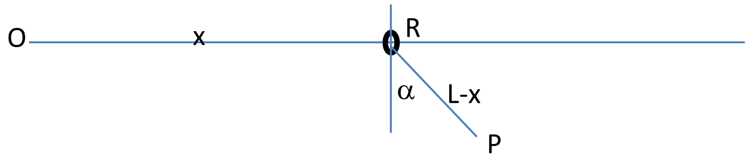
$$d^2 = x^2 + \left(\frac{u^2}{2g} - \frac{gx^2}{2u^2}\right)^2 = \left(\frac{u^2}{2g} + \frac{gx^2}{2u^2}\right)^2$$ **M1 A1**

Therefore the maximum value must occur when x is maximised (when $x = \frac{u^2}{g}$) **M1**

The maximum value for d is $\frac{u^2}{g}$ **A1**

B1	For writing down the equation for the horizontal motion
M1	For attempting to write down the equation for the vertical motion
A1	For the correct equation
M1	For attempting to eliminate t
A1	For the correct equation
M1	For $\sec^2\theta = 1 + \lambda^2$
A1	For the final answer
M1	For attempting to complete the square (or for differentiation)
A1	For the correct expression (or correct derivative)
M1	For identifying the maximum value (either setting derivative to 0 or noting the value of x for which it occurs in the completed square form)
A1	For obtaining the correct maximum value
B1	For explaining that all values up to this are possible
B1	Quadratic shape to graph
B1	Maximum at point $\left(0, \frac{u^2}{2g}\right)$
B1	Intersection with x axis at $\left(\frac{u^2}{g}, 0\right)$
B1	Region bounded by curve and axes identified
M1	For trying to find an expression for d^2
A1	For the correct simplified expression
M1	For observing that the greatest value of x maximises this
A1	For the correct answer (note that this must be supported by the previous marks as candidates may misinterpret the question as asking for the range)

Question 11



Coordinates of P:

$$(x + (L - x)\sin\alpha\,, -(L - x)\cos\alpha)$$ **B1 B1**

(i) The resultant force acting on P is:

$$T\cos\alpha - kmg$$

Upwards **M1 A1**

The acceleration of P vertically is:

$$\ddot{x}\cos\alpha$$

upwards **B1**

Therefore, by Newton's 2nd Law: **M1 A1**

$$T\cos\alpha - kmg = km\ddot{x}\cos\alpha$$ **(AG)**

Horizontally:

The resultant force on P is:

$$T\sin\alpha$$

To the left.

The acceleration of P horizontally is:

$$(1 - \sin\alpha)\ddot{x}$$

To the right. **B1 B1**

Newton's 2nd Law gives:

$$T\sin\alpha = -km(1 - \sin\alpha)\ddot{x}$$ **M1 A1**

Similarly, for R:

$$T - T\sin\alpha = -m\ddot{x}$$ **M1 A1**

(ii) Multipliying the last equation by $\sin\alpha$ gives:

$$T\sin\alpha\,(1 - \sin\alpha) = -mx\,\mathrm{s\ddot{\imath}n}\,\alpha$$

And so (substituting the equation from the horizontal motion of P):

$$-km(1 - \sin\alpha)\ddot{x}(1 - \sin\alpha) = -mx\,\mathrm{s\ddot{\imath}n}\,\alpha$$

Which simplifies to:

$$k = \frac{\sin\alpha}{(1 - \sin\alpha)^2}$$ **M1 A1 (AG)**

Sketch of $y = \frac{x}{(1-x)^2}$ shows a continuous curve through the origin with an asymptote at $x = 1$, therefore for any value of k there is a possible value between 0 and 1 for $\sin\alpha$. **B1 B1**

(iii) We already have that:

$$T\cos\alpha = kmg + km\ddot{x}\cos\alpha$$

And

$$T\sin\alpha = -km(1-\sin\alpha)\ddot{x}$$

Dividing the second equation by the first (to eliminate T):

$$\tan\alpha = \frac{-(1-\sin\alpha)\ddot{x}}{g+\ddot{x}\cos\alpha}$$ **M1**

Therefore:

$$g\tan\alpha + \ddot{x}\sin\alpha = -\ddot{x} + \ddot{x}\sin\alpha$$

And so **M1 A1**

$$\ddot{x} = -g\tan\alpha$$ **(AG)**

B1	For the x coordinate
B1	For the y coordinate
(i)	
M1	For resolving forces at P vertically
A1	For the correct resultant force vertically
B1	For finding the acceleration of P vertically with sign consistent with resultant force
M1	For applying Newton's 2^{nd} Law
A1	For reaching the given result
B1	For the correct component of the resultant force on P
B1	For the acceleration of P horizontally (ignore inconsistent directions here)
M1	For Newton's 2^{nd} Law (with directions consistent)
A1	For the correct equation
M1	For obtaining the resultant force on R
A1	For obtaining the equation relating to the motion of R
(ii)	
M1	For attempting to eliminate T from the equations
A1	For reaching the given formula
B1	Sketch of the graph with asymptote shown at x=1
B1	Explanation that this means that motion with α constant is always possible
(iii)	
M1	For eliminating T from the two equations
M1	For rearranging to make $\ddot{x}$ the subject
A1	For reaching the given answer

Question 12

(i) The required probability is given by:

$$\frac{P(t < T < t + \delta t)}{P(T > t)} = \frac{F(t + \delta t) - F(t)}{1 - F(t)}$$ **M1 A1**

For small values of δt:

$$\frac{F(t + \delta t) - F(t)}{\delta t} \approx f(t)$$ **M1**

Therefore the required probability is given by:

$$\frac{f(t)\delta t}{1 - F(t)} = h(t)\delta t$$ **A1 (AG)**

(ii)

$$f(t) = \frac{1}{a}$$ **B1**

Therefore

$$h(t) = \frac{1/a}{1 - t/a} = \frac{1}{a - t}$$ **A1**

Sketch of graph for h(t) and f(t) **B1**

B1

B1

$$\frac{F'(t)}{1 - F(t)} = \frac{1}{t}$$ **B1**

Integrating with respect to t:

$$-\ln|1 - F(t)| = \ln|kt|$$

So

$$F(t) = 1 - \frac{A}{t}$$ **M1 A1**

Since $F(t) = 0$, $A = a$. **B1**

Therefore

$$f(t) = \frac{a}{t^2}$$ **B1**

(iv) If $h(t)$ is constant (=k) for $t > b$ then integrating t equation for h(t) gives:

$$-\ln|1 - F(t)| = kt + c$$

Therefore

$$F(t) = 1 - e^{-kt-c}$$ **M1**

And so (since F(b)=0)

$$F(t) = 1 - e^{-k(t-b)}$$ **B1**

Differentiating gives:

$$f(t) = ke^{-k(t-b)}$$ **A1 (AG)**

If $f(t) = ke^{-k(t-b)}$, then $h(t)$ can be calculated from the formula and satisfies the given conditions **B1**

(v) As in previous two parts integration gives:

$$-\ln|1 - F(t)| = \left(\frac{t}{\theta}\right)^{\lambda}$$

So

$$F(t) = 1 - e^{-\left({t}/{\theta}\right)^{\lambda}}$$ **M1**

Therefore

$$f(t) = \frac{\lambda x^{\lambda-1}}{\theta^{\lambda}} e^{-\left({t}/{\theta}\right)^{\lambda}}$$ **A1**

(i)	
M1	For expressing the conditional probability as a quotient of two probabilities
A1	For expressing in terms of the cumulative distribution function
M1	For identifying the relationship for small values of δt
A1	For reaching the given expression
(ii)	
B1	For differentiating to find f(t)
A1	For obtaining the equation for h(t)
B1	For graph of f(t) (Horizontal line at height 1/a between x=0 and x=a
B1	For graph of h(t) (correct shape)
B1	For asymptote at x=a
(iii)	
B1	For writing down the differential equation
M1	Integration of one side correct
A1	Correct function for F(x)
B1	Evaluate constant from integration
B1	Differentiate for f(x)
(iv)	
M1	For integrating the differential equation to reach F(t)
B1	For evaluating the constant
A1	For differentiating to get f(t)
B1	For checking that the conditions on h(t) are satisfied if f(t) is the given function
(v)	
M1	For integrating the differential equation to reach F(t)
A1	For differentiating to get f(t)

Question 13

For X=4, the following needs to happen:

The second number is different from the first (probability $1-\frac{1}{n}$)

The third number is different from the first two (probability $1-\frac{2}{n}$) **B1**

The fourth number is the same as one of the first three (probability $\frac{3}{n}$) **M1**

Therefore

$$P(X=4)=\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{3}{n}$$ **A1**

(i) By a similar argument to the one above:

$$P(X=r)=\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-2}{n}\right)\frac{r-1}{n}$$ **B1**

Since all possibilities are covered by $2\le r\le n+1$ and all probabilities sum to 1:

$$\frac{1}{n}+\left(1-\frac{1}{n}\right)\frac{2}{n}+\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{3}{n}+\cdots=1$$ **B1 (AG)**

(ii)

$$E(X)=\frac{2}{n}+3\left(1-\frac{1}{n}\right)\frac{2}{n}+4\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\frac{3}{n}+\cdots$$
$$+(n+1)\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{n-1}{n}\right)$$ **M1 A1**

(iii) For $X\ge k$, the numbers must all be different up to and including the (k-1)th not matching any of the previous (k-2)

$$P(X\ge k)=\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{k-2}{n}\right)$$ **B1 M1 A1**

(iv) For any random variable taking the values 1 to N:

$$E(Y) = \sum_{k=1}^{N} kP(Y = k)$$

So

$$E(Y) = \begin{array}{ccccccccc} P(Y=1) & + & P(Y=2) & + & P(Y=3) & + & \cdots & + & P(Y=N) \\ & & P(Y=2) & + & P(Y=3) & + & \cdots & + & P(Y=N) \\ & & & & P(Y=3) & + & \cdots & + & P(Y=N) \end{array}$$

Etc.

Therefore

$$E(Y) = \sum_{k=1}^{N} P(Y \geq k)$$

B1
M1 A1
(AG)

Therefore, combining the answers to (ii) and (iii):

$$\frac{2}{n} + 3\left(1 - \frac{1}{n}\right)\frac{2}{n} + 4\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots$$
$$+ (n+1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$
$$= 1 + 1 + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)$$
$$+ \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$

M1 A1
A1

Subtract $\frac{1}{n} + \left(1 - \frac{1}{n}\right)\frac{2}{n} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots$ from the LHS and 1 from the RHS (they are equal by part (i))

$$\frac{1}{n} + 2\left(1 - \frac{1}{n}\right)\frac{2}{n} + 3\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots + n\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$
$$= 1 + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)$$
$$+ \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$

M1 A1

Rearranging:

$$0 = 1 - \frac{1}{n} + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2^2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\left(1 - \frac{3^2}{n}\right)$$

M1 A1
(AG)

B1	Identifying the probability that the second number is different from the first
M1	Identifying the sequence of events to give X=4 (possibly by drawing a tree diagram)
A1	Correct probability
(i)	
B1	Correct probability
B1	Justification of the given equation
(ii)	
M1	Use of the correct formula for E(X)
A1	Correct expression for E(X) (note – no simplification is required)
(iii)	
B1	For identifying that the probability will be the product of the first (k-2) probabilities in the tree diagram
M1	For a correct product (award this mark even if the number of factors is wrong as long as they are of the correct form)
A1	For the correct probability
(iv)	
B1	For writing down the formula for E(Y)
M1	For regrouping the sum into cumulative probabilities
A1	For reaching the given answer
M1	For putting together the results from parts (ii) and (iii)
A1	For including one of the 1s at the start of the RHS (either the case k=1 or k=2)
A1	For including both of the 1s at the start of the RHS
M1	For applying the result of part (i)
A1	For reaching the equation
M1	For grouping all terms on the same side
A1	For reaching the required equation