1. $t = \tan\frac{1}{2}x$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{1}{2}x = \frac{1}{2}\left(1+\tan^2\frac{1}{2}x\right) = \frac{1}{2}(1+t^2)$ **(*)** **M1 $\sec^2$** **M1 $1+\tan^2$**

$$\sin x = 2\sin\frac{1}{2}x\cos\frac{1}{2}x = 2\frac{\sin\frac{1}{2}x}{\cos\frac{1}{2}x}\cos^2\frac{1}{2}x = 2\tan\frac{1}{2}x\frac{1}{\sec^2\frac{1}{2}x}$$

$$= 2\tan\frac{1}{2}x\frac{1}{\left(1+\tan^2\frac{1}{2}x\right)} = \frac{2t}{1+t^2}$$

(*) M1 $\sin 2A$ M1 $\cos^2 = 1/\sec^2$ A1 both correct (5)

$$\int_0^{\frac{1}{2}\pi}\frac{1}{1+a\sin x}\,dx = \int_0^1\frac{1}{1+a\frac{2t}{1+t^2}}\frac{1}{\frac{1}{2}(1+t^2)}\,dt = 2\int_0^1\frac{1}{1+2at+t^2}dt$$

M1 full substitution for x, and dx **A1 fully simplified (condone incorrect limits)**

$$= 2\int_0^1\frac{1}{(1-a^2)+(t+a)^2}dt$$

Using $t+a = \sqrt{1-a^2}\tan u$, $\frac{dt}{du} = \sqrt{1-a^2}\sec^2 u$,

so

$$\int_0^{\frac{1}{2}\pi}\frac{1}{1+a\sin x}\,dx = 2\int_{\tan^{-1}\frac{a}{\sqrt{1-a^2}}}^{\tan^{-1}\frac{1+a}{\sqrt{1-a^2}}}\frac{1}{(1-a^2)+(1-a^2)\tan^2 u}\sqrt{1-a^2}\sec^2 u\,du$$

M1 full substitution including limits

$$= 2\int_{\tan^{-1}\frac{a}{\sqrt{1-a^2}}}^{\tan^{-1}\frac{1+a}{\sqrt{1-a^2}}}\frac{1}{\sqrt{1-a^2}}\,du = \frac{2}{\sqrt{1-a^2}}\,[u]_{\tan^{-1}\frac{a}{\sqrt{1-a^2}}}^{\tan^{-1}\frac{1+a}{\sqrt{1-a^2}}}$$

or alternatively $2\int_0^1\frac{1}{(1-a^2)+(t+a)^2}dt = 2\left[\frac{1}{\sqrt{1-a^2}}\tan^{-1}\frac{t+a}{\sqrt{1-a^2}}\right]_0^1$, or using a substitution for $t+a$

$$= \frac{2}{\sqrt{1-a^2}}\left(\tan^{-1}\frac{1+a}{\sqrt{1-a^2}} - \tan^{-1}\frac{a}{\sqrt{1-a^2}}\right)$$

M1 integration and evaluation A1

$$= \frac{2}{\sqrt{1-a^2}}\tan^{-1}\left(\tan\left(\tan^{-1}\frac{1+a}{\sqrt{1-a^2}} - \tan^{-1}\frac{a}{\sqrt{1-a^2}}\right)\right)$$

$$= \frac{2}{\sqrt{1-a^2}}\tan^{-1}\left(\frac{\frac{1+a}{\sqrt{1-a^2}} - \frac{a}{\sqrt{1-a^2}}}{1+\frac{1+a}{\sqrt{1-a^2}}\frac{a}{\sqrt{1-a^2}}}\right)$$

M1 correct use of compound angle formula

$$= \frac{2}{\sqrt{1-a^2}} \tan^{-1}\left(\frac{\frac{1}{\sqrt{1-a^2}}}{1 + \frac{a+a^2}{1-a^2}} \right)$$

$$= \frac{2}{\sqrt{1-a^2}} \tan^{-1}\left(\frac{\sqrt{1-a^2}}{1+a} \right) = \frac{2}{\sqrt{1-a^2}} \tan^{-1}\left(\frac{\sqrt{(1-a)(1+a)}}{1+a} \right) = \frac{2}{\sqrt{1-a^2}} \tan^{-1} \frac{\sqrt{1-a}}{\sqrt{1+a}}$$

(*) A1 **(7)**

$$I_{n+1} + 2I_n = \int_0^{\frac{1}{2}\pi} \frac{\sin^{n+1} x + 2\sin^n x}{2 + \sin x} dx = \int_0^{\frac{1}{2}\pi} \frac{\sin^n x\,(\sin x + 2)}{2 + \sin x}\, dx = \int_0^{\frac{1}{2}\pi} \sin^n x\, dx$$

B1

$$I_3 + 2I_2 = \int_0^{\frac{1}{2}\pi} \sin^2 x\, dx = \int_0^{\frac{1}{2}\pi} \frac{1 - \cos 2x}{2} dx = \left[\frac{1}{2}x - \frac{1}{4}\sin 2x\right]_0^{\frac{1}{2}\pi} = \frac{1}{4}\pi$$

M1 using cos 2x correctly **A1**

$$I_2 + 2I_1 = \int_0^{\frac{1}{2}\pi} \sin x\, dx = [-\cos x]_0^{\frac{1}{2}\pi} = 1$$

$$I_1 + 2I_0 = \int_0^{\frac{1}{2}\pi} 1\, dx = [x]_0^{\frac{1}{2}\pi} = \frac{1}{2}\pi$$

B1 for getting both

$$I_0 = \int_0^{\frac{1}{2}\pi} \frac{1}{2 + \sin x}\, dx = \frac{1}{2}\int_0^{\frac{1}{2}\pi} \frac{1}{1 + \frac{1}{2}\sin x}\, dx = \frac{1}{2}\, \frac{2}{\sqrt{1 - \frac{1}{4}}}\, \tan^{-1} \frac{\sqrt{1 - \frac{1}{2}}}{\sqrt{1 + \frac{1}{2}}} = \frac{2}{\sqrt{3}}\, \frac{\pi}{6} = \frac{\pi}{3\sqrt{3}}$$

M1 to use previous part **A1**

$$I_3 = \frac{1}{4}\pi - 2\left(1 - 2\left(\frac{1}{2}\pi - 2\frac{\pi}{3\sqrt{3}}\right)\right) = \frac{1}{4}\pi - 2 + 2\pi - \frac{8\pi}{3\sqrt{3}} = \left(\frac{9}{4} - \frac{8}{3\sqrt{3}}\right)\pi - 2$$

M1 combining all **A1** **(8)**

2. $y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$

$\sqrt{1-x^2}\, y = \sin^{-1} x$

$\sqrt{1-x^2}\,\frac{dy}{dx} - \frac{x}{\sqrt{1-x^2}}\, y = \frac{1}{\sqrt{1-x^2}}$

$(1-x^2)\frac{dy}{dx} - xy = 1$ (*)

M1 A1 for use of product rule **M1 A1 algebraic simplification** **(4)**

Alternatively,

$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}}$

$\frac{dy}{dx} = \frac{\sqrt{1-x^2}\frac{1}{\sqrt{1-x^2}} - \sin^{-1} x \frac{-x}{\sqrt{1-x^2}}}{1-x^2}$ **M1 A1 for quotient rule**

$= \frac{1+x\frac{\sin^{-1} x}{\sqrt{1-x^2}}}{1-x^2} = \frac{1+xy}{1-x^2}$ **M1 A1 algebraic simplification** **(4)**

Alternatively,

$y = \frac{\sin^{-1} x}{\sqrt{1-x^2}} = \sin^{-1} x\ (1-x^2)^{-\frac{1}{2}}$

$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}\ (1-x^2)^{-\frac{1}{2}} + \sin^{-1} x\ x(1-x^2)^{-\frac{3}{2}}$ **M1 A1 for use of product rule**

then M1 A1 algebraic simplification as before to obtain required result **(4)**

Suppose $(1-x^2)\frac{d^{k+2}y}{dx^{k+2}} - (2k+3)x\frac{d^{k+1}y}{dx^{k+1}} - (k+1)^2\ \frac{d^k y}{dx^k} = 0$ for some particular positive integer *k* **E1**

Then $(1-x^2)\frac{d^{k+3}y}{dx^{k+3}} - 2x\frac{d^{k+2}y}{dx^{k+2}} - (2k+3)x\frac{d^{k+2}y}{dx^{k+2}} - (2k+3)\frac{d^{k+1}y}{dx^{k+1}} - (k+1)^2\ \frac{d^{k+1}y}{dx^{k+1}} = 0$

$(1-x^2)\frac{d^{k+3}y}{dx^{k+3}} - (2k+5)x\frac{d^{k+2}y}{dx^{k+2}} - (k^2+4k+4)\frac{d^{k+1}y}{dx^{k+1}} = 0$

$(1-x^2)\frac{d^{k+3}y}{dx^{k+3}} - (2(k+1)+3)x\frac{d^{k+2}y}{dx^{k+2}} - \left((k+1)+1\right)^2\frac{d^{k+1}y}{dx^{k+1}} = 0$

Which is the required result for *k*+1 **M1 A1**

As $(1-x^2)\frac{dy}{dx} - xy = 1$

$(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} - x\frac{dy}{dx} - y = 0$ **M1**

$(1-x^2)\frac{d^2y}{dx^2} - 3x\frac{dy}{dx} - y = 0$ which is the result for $n = 0$ **A1**

Hence, by PMI, the result is true for non-negative integer *n*, and thus for positive integer *n*. **E1 (6)**

or $(1-x^2)\frac{d^3y}{dx^3}-2x\frac{dy}{dx}-3x\frac{dy}{dx}-3\frac{dy}{dx}-\frac{dy}{dx}=0$ which is the result for $n=1$ case

Alternatively, $(1-x^2)\frac{dy}{dx}-xy=1$,

Differentiating *n+1* times by Leibnitz

$(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}+(n+1)(-2x)\frac{d^{n+1}y}{dx^{n+1}}+\frac{(n+1)n}{2}(-2)\frac{d^ny}{dx^n}-x\frac{d^{n+1}y}{dx^{n+1}}+(n+1)(-1)\frac{d^ny}{dx^n}=0$ **M1 A4**

so $(1-x^2)\frac{d^{n+2}y}{dx^{n+2}}-(2n+3)x\frac{d^{n+1}y}{dx^{n+1}}-(n+1)^2\frac{d^ny}{dx^n}=0$ **A1**

$y=y(0)+xy'(0)+\frac{x^2}{2}y''(0)+\frac{x^3}{3!}y'''(0)+\cdots$

$y(0)=\frac{\sin^{-1}0}{\sqrt{1-0^2}}=0$, $(1-0^2)y'(0)-0y(0)=1$ so $y'(0)=1$ **M1**

$(1-0^2)y''(0)-3.0y'(0)-y(0)=0$ so $y''(0)=0$ **M1**

$(1-0^2)y'''(0)-5.0y''(0)-4y'(0)=0$ so $y'''(0)=2^2$

Similarly $y''''(0)=0$, $y'''''(0)=4^22^2$

So in general $y^{(2r)}(0)=0$ **A1** and $y^{(2r+1)}(0)=2^{2r}(r!)^2$ **A1**

Thus, in the Maclaurin series, the general term for even powers of *x* is zero, and for odd powers of *x* is $2^{2r}(r!)^2\frac{x^{2r+1}}{(2r+1)!}=\frac{2^{2r}(r!)^2}{(2r+1)!}x^{2r+1}$ or alternatively $\frac{2^{n-1}\left(\left(\frac{n-1}{2}\right)!\right)^2}{n!}x^n$ **M1 A1 (6)**

$y=0+x+0+\frac{2^2}{3!}x^3+0+\frac{4^22^2}{5!}x^5+0+\cdots$

$\frac{y}{x}=1+\frac{2^2}{3!}x^2+\frac{4^22^2}{5!}x^4+\cdots$ **M1**

So if $x=\frac{1}{2}$, **M1** $y=\frac{\sin^{-1}\frac{1}{2}}{\sqrt{1-\frac{1}{4}}}=\frac{2}{\sqrt{3}}\frac{\pi}{6}=\frac{\pi}{3\sqrt{3}}$ **A1** and thus

$\frac{2\pi}{3\sqrt{3}}=1+\frac{1}{3!}+\frac{2^2}{5!}x^4+\frac{3^22^2}{7!}+\cdots+\frac{n^2\ldots3^22^2}{(2n+1)!}+\cdots$ **A1 (4)**

3. $p_i.\sum_{r=1}^{4} p_r = p_i.0 = 0$ **M1**

So $p_i.p_i + p_i.p_j + p_i.p_k + p_i.p_l = 0$

By symmetry, $p_i.p_j = p_i.p_k = p_i.p_l$ where $i \neq j, i \neq k, i \neq l$ **M1** and $p_i.p_i = 1$ **B1**

So $1 + 3p_i.p_j = 0$, and thus $p_i.p_j = -\frac{1}{3}$ **(*) A1** **(4)**

(i) $\sum_{i=1}^{4}(XP_i)^2 = \sum_{i=1}^{4}(p_i - x).(p_i - x) = \sum_{i=1}^{4}(p_i.p_i - 2x.p_i + x.x) = \sum_{i=1}^{4}(1 - 2x.p_i + 1)$

M1 **A1**

$= \sum_{i=1}^{4}(2 - 2x.p_i) = 8 - 2x.\sum_{i=1}^{4} p_i = 8 - 2x.0 = 8$

M1 **(*) A1** **(4)**

(ii) $p_1.p_2 = -\frac{1}{3}$ so $0.a + 0.0 + 1.b = -\frac{1}{3}$ and thus $b = -\frac{1}{3}$

$p_2.p_2 = 1$ so $a.a + 0.0 + b.b = 1$ and thus $a^2 + \frac{1}{9} = 1$, $a^2 = \frac{8}{9}$, $a = \pm\frac{2\sqrt{2}}{3}$ and as *a* is positive, $a = \frac{2\sqrt{2}}{3}$ **(*) M1 A1**

If $P_3 = (c, d, e)$ and $P_4 = (f, g, h)$, as $p_1.p_j = -\frac{1}{3}$ for $j \neq 1$, $e = h = -\frac{1}{3}$ **B1**

As $p_2.p_3 = -\frac{1}{3}$, $\frac{2\sqrt{2}}{3}.c + 0.d + -\frac{1}{3}.-\frac{1}{3} = -\frac{1}{3}$, $\frac{2\sqrt{2}}{3}.c = -\frac{4}{9}$, $c = -\frac{\sqrt{2}}{3}$

But $p_3.p_3 = 1$ so $c^2 + d^2 + e^2 = 1$, i.e. $\frac{2}{9} + d^2 + \frac{1}{9} = 1$, $d = \pm\frac{\sqrt{2}}{\sqrt{3}}$ **M1 A1(c) A1(d)**

So $P_3, P_4 = \left(-\frac{\sqrt{2}}{3}, \pm\frac{\sqrt{2}}{\sqrt{3}}, -\frac{1}{3}\right)$ **(6)**

(iii) (i) $\sum_{i=1}^{4}(XP_i)^4 = \sum_{i=1}^{4}\left((p_i - x).(p_i - x)\right)^2 = \sum_{i=1}^{4}(2 - 2x.p_i)^2 = 4\sum_{i=1}^{4}(1 - x.p_i)^2$

M1 **A1**

$= 4\sum_{i=1}^{4}(1 - 2x.p_i + (x.p_i)^2) = 16 - 8x.\sum_{i=1}^{4} p_i + 4\sum_{i=1}^{4}(x.p_i)^2$ **M1**

$= 16 - 0 + 4\left(z^2 + \left(\frac{2\sqrt{2}}{3}x - \frac{1}{3}z\right)^2 + \left(-\frac{\sqrt{2}}{3}x + \frac{\sqrt{2}}{\sqrt{3}}y - \frac{1}{3}z\right)^2 + \left(-\frac{\sqrt{2}}{3}x - \frac{\sqrt{2}}{\sqrt{3}}y - \frac{1}{3}z\right)^2\right)$ **M1**

$= 16 + 4\left(\frac{4}{3}x^2 + \frac{4}{3}y^2 + \frac{4}{3}z^2\right) = \frac{64}{3}$ which is independent of the position of X.

A1 **A1** (actual value not required, merely independence so may stop with unsimplified result) **(6)**

4. $\left(z-e^{i\theta}\right)\left(z-e^{-i\theta}\right)=z^2-z\left(e^{i\theta}+e^{-i\theta}\right)+1=z^2-z(\cos\theta+i\sin\theta+\cos\theta-i\sin\theta)+1$

M1 **M1**

$=z^2-2z\cos\theta+1$ **(*) A1** **(3)**

If $e^{i\theta}$ is a 2*n*th root of -1 then $e^{i2n\theta}=-1=e^{i\pi+2m\pi}$ where $-n\le m\le n-1$

Therefore $\theta=\frac{2m+1}{2n}\pi$ and so the roots are $e^{i\frac{2m+1}{2n}\pi}$, $-n\le m\le n-1$ **B1, B1** **(2)**

The factors of $z^{2n}+1$ are $\left(z-e^{i\frac{2m+1}{2n}\pi}\right)$, $-n\le m\le n-1$ **M1**

So $z^{2n}+1=\left(z-e^{i\frac{1}{2n}\pi}\right)\left(z-e^{-i\frac{1}{2n}\pi}\right)\left(z-e^{i\frac{3}{2n}\pi}\right)\left(z-e^{-i\frac{3}{2n}\pi}\right)\cdots\left(z-e^{i\frac{2n-1}{2n}\pi}\right)\left(z-e^{-i\frac{2n-1}{2n}\pi}\right)$

M1

$=\left(z^2-2z\cos\frac{1}{2n}\pi+1\right)\left(z^2-2z\cos\frac{3}{2n}\pi+1\right)\cdots\left(z^2-2z\cos\frac{2n-1}{2n}\pi+1\right)$

$=\prod_{k=1}^{n}\left(z^2-2z\cos\frac{2k-1}{2n}\pi+1\right)$ **(*) A1** **(3)**

(i) If $z=i$ and *n* is even, $z^{2n}+1=1+1=2$ **B1** and

$$\prod_{k=1}^{n}\left(z^2-2z\cos\frac{2k-1}{2n}\pi+1\right)=\prod_{k=1}^{n}\left(-2i\cos\frac{2k-1}{2n}\pi\right)=(-2i)^n\prod_{k=1}^{n}\left(\cos\frac{2k-1}{2n}\pi\right)$$

B1

$=(-1)^n2^n(-1)^{\frac{n}{2}}\cos\frac{\pi}{2n}\cos\frac{3\pi}{2n}\cos\frac{5\pi}{2n}\cdots\cos\frac{2n-1}{2n}\pi$ **M1**

i.e. $\cos\frac{\pi}{2n}\cos\frac{3\pi}{2n}\cos\frac{5\pi}{2n}\cdots\cos\frac{2n-1}{2n}\pi=(-1)^{\frac{n}{2}}2^{1-n}$ **(*) A1** **(4)**

(ii)

$$1+z^{2n}=\prod_{k=1}^{n}\left(z^2-2z\cos\frac{2k-1}{2n}\pi+1\right)$$

But $1+z^{2n}=(1+z^2)(1-z^2+z^4-\cdots+z^{2n-2})$ if *n* is odd.

So $(1+z^2)(1-z^2+z^4-\cdots+z^{2n-2})=\left(z^2-2z\cos\frac{1}{2n}\pi+1\right)\left(z^2-2z\cos\frac{3}{2n}\pi+1\right)\cdots\left(z^2-\right.$
$\left.2z\cos\frac{n-2}{2n}\pi+1\right)\left(z^2-2z\cos\frac{n}{2n}\pi+1\right)\left(z^2-2z\cos\frac{n+2}{2n}\pi+1\right)\cdots\left(z^2-2z\cos\frac{2n-1}{2n}\pi+1\right)$

this term $=z^2+1$ **B1**

Thus $(1-z^2+z^4-\cdots+z^{2n-2})=\left(z^2-2z\cos\frac{1}{2n}\pi+1\right)\left(z^2-2z\cos\frac{3}{2n}\pi+1\right)\cdots\left(z^2-\right.$
$\left.2z\cos\frac{n-2}{2n}\pi+1\right)\left(z^2-2z\cos\frac{n+2}{2n}\pi+1\right)\cdots\left(z^2-2z\cos\frac{2n-1}{2n}\pi+1\right)$

If $z = i$ and n is odd,

$1 - z^2 + z^4 - \cdots + z^{2n-2} = 1 - i^2 + i^4 - \cdots + i^{2n-2} = 1 + 1 + \cdots + (-1)^{n-1} = n$ **B1**

and

$\left(z^2 - 2z\cos\frac{1}{2n}\pi + 1\right)\left(z^2 - 2z\cos\frac{3}{2n}\pi + 1\right)\cdots\left(z^2 - 2z\cos\frac{n-2}{2n}\pi + 1\right)\left(z^2 - 2z\cos\frac{n+2}{2n}\pi + 1\right)\cdots\left(z^2 - 2z\cos\frac{2n-1}{2n}\pi + 1\right)$

$= \left(-2i\cos\frac{1}{2n}\pi\right)\left(-2i\cos\frac{3}{2n}\pi\right)\cdots\left(-2i\cos\frac{n-2}{2n}\pi\right)\left(-2i\cos\frac{n+2}{2n}\pi\right)\cdots\left(-2i\cos\frac{2n-1}{2n}\pi\right)$ **M1**

$= (-2i)^{\,n-1}\left(\cos\frac{1}{2n}\pi\right)\left(\cos\frac{3}{2n}\pi\right)\cdots\left(\cos\frac{n-2}{2n}\pi\right)\left(-\cos\frac{n-2}{2n}\pi\right)\cdots\left(-\cos\frac{1}{2n}\pi\right)$ **M1**

$= (-2i)^{\,n-1}(-1)^{\frac{n-1}{2}}\left(\cos\frac{1}{2n}\pi\right)^2\left(\cos\frac{3}{2n}\pi\right)^2\cdots\left(\cos\frac{n-2}{2n}\pi\right)^2$ **A1**

$= (-1)^{n-1}2^{n-1}(-1)^{\frac{n-1}{2}}(-1)^{\frac{n-1}{2}}\left(\cos\frac{1}{2n}\pi\right)^2\left(\cos\frac{3}{2n}\pi\right)^2\cdots\left(\cos\frac{n-2}{2n}\pi\right)^2$ **A1**

$= 2^{n-1}\left(\cos\frac{1}{2n}\pi\right)^2\left(\cos\frac{3}{2n}\pi\right)^2\cdots\left(\cos\frac{n-2}{2n}\pi\right)^2$ **A1**

So $\left(\cos\frac{\pi}{2n}\right)^2\left(\cos\frac{3\pi}{2n}\right)^2\left(\cos\frac{5\pi}{2n}\right)^2\cdots\left(\cos\frac{(n-2)\pi}{2n}\right)^2 = n.2^{-(n-1)}$ **(*) A1 (8)**

5. (i) $q^n N = qq^{n-1}N = p^n$ **E1**

p divides p^n, so p divides $qq^{n-1}N$ and as p and q are coprime, p divides $q^{n-1}N$ **E1**

Repeating this argument, p divides $q^{n-2}N$, etc. so , p divides N . **E1** Letting $N = pQ_1$, we have $q^n pQ_1 = p^n$ and so $q^n Q_1 = p^{n-1}$. **E1** The previous argument yields $Q_1 = pQ_2$ etc. so $N = p^n Q_n$ or in in other words $N = kp^n$ as required. **E1** **(5)**

So as $q^n N = p^n$, $q^n kp^n = p^n$, that is $q^n k = 1$ and as q and k are positive integers they must both be 1 . **E1**

Thus if $\sqrt[n]{N} = \frac{p}{q}$ where p and q are coprime, i.e. if it is rational, it can be written in lowest terms, then $q^n N = p^n$ and so $q = 1$ and thus $\sqrt[n]{N}$ is an integer. **E1** Otherwise, $\sqrt[n]{N}$ cannot be written as $\frac{p}{q}$ with p and q are coprime, that is, it is irrational. **E1** **(3)**

(ii) As a and b are coprime, and b^a divides $a^a d^b$, by the same reasoning used in part (i), b^a divides d^b. So $d^b = kb^a$, for some integer k . **E1**

As $a^a d^b = b^a c^b$, $a^a k b^a = b^a c^b$, so $a^a k = c^b$. **E1**

As c and d are coprime, and c^b divides $a^a d^b$, by the same reasoning used in part (i), c^b divides a^a. So $a^a = k' c^b$, for some integer k' , **E1** so $k' c^b k = c^b$, **E1** and thus $k'k = 1$, i.e. $k = k' = 1$, and so $d^b = b^a$. **E1** **(5)**

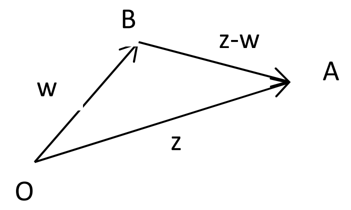
If p is a prime factor of d , then p divides d^b , so p divides b^a. **E1** $b^a = bb^{a-1}$ so if p does not divide b , p divides b^{a-1} by assumed result, and repetition of this argument leads to a contradiction. So p is a prime factor of b . **E1** **(2)**

If p^m is the highest power of p that divides d , then p^{mb} is the highest power of p that divides d^b . Similarly, p^{na} is the highest power of p that divides b^a . **E1** Thus as $d^b = b^a$, $mb = na$, and so $b = \frac{na}{m}$. **B1**

p^n divides b , so p^n divides $\frac{na}{m}$, so p^n divides na . But a and b are coprime so p^n divides n and thus $p^n \leq n$. **E1** As $y^x > x$ for $y \geq 2$ if $x > 0$, then $p = 1$. Thus b is only divisible by 1 so $b = 1$. **E1**

$a^a d^b = b^a c^b \Rightarrow \frac{a^a}{b^a} = \frac{c^b}{d^b} \Rightarrow \left(\frac{a}{b}\right)^{\frac{a}{b}} = \frac{c}{d}$ Thus if r is a positive rational $\frac{a}{b}$, such that $r^r = \frac{c}{d}$ is rational then $b = 1$ so r is a positive integer. **E1** **(5)**

6.



B1

$AB \leq OA + OB$ (Triangle inequality) **B1**

$|z - w| \leq |z| + |w|$ **(*) B1** **(3)**

(i) $|z - w|^2 = (z - w)(z - w)^* = (z - w)(z^* - w^*)$ **M1** use of conjugate & algebra of it

$= zz^* - wz^* - zw^* + ww^* = |z|^2 + |w|^2 - (E - 2|zw|)$ **M1** algebra and substitution

$= |z|^2 + |w|^2 + 2|zw| - E = (|z| + |w|)^2 - E$ **(*) A1**

$|z - w|^2$ is real, $(|z| + |w|)^2$ is real, so E is real. **E1**

As $|z - w| \leq |z| + |w|$, $|z - w|^2 \leq (|z| + |w|)^2$, so $E = (|z| + |w|)^2 - |z - w|^2 \geq 0$ as required.

E1 (5)

(ii) $|1 - zw^*|^2 = (1 - zw^*)(1 - zw^*)^* = (1 - zw^*)(1 - z^*w)$ **M1** as before

$= 1 - zw^* - z^*w + zz^*ww^* = 1 + |z|^2|w|^2 - (E - 2|zw|)$ **M1** as before

$= 1 + 2|zw| + |zw|^2 - E = (1 + |zw|)^2 - E$ **(*) A1 (3)**

As $E \geq 0$, $|z| > 1$, and $|w| > 1$, $E(1 - |z|^2)(1 - |w|^2) \geq 0$ **M1**

Thus $E(1 + 2|zw| + |zw|^2 - |z|^2 - 2|zw| - |w|^2) \geq 0$ **M1**

Therefore $E((1 + |z||w|)^2 - (|z| + |w|)^2) \geq 0$ **M1**

Hence $-E(|z| + |w|)^2 \geq -E(1 + |z||w|)^2$, **M1**

and so $(1 + |zw|)^2(|z| + |w|)^2 - E(|z| + |w|)^2 \geq (1 + |zw|)^2(|z| + |w|)^2 - E(1 + |z||w|)^2$ **M1**

and $\frac{(|z|+|w|)^2}{(1+|z||w|)^2} \geq \frac{(|z|+|w|)^2-E}{(1+|zw|)^2-E} = \frac{|z-w|^2}{|1-zw^*|^2}$ **A1**

As all terms are squares of positive expressions, we can square root, to give $\frac{|z-w|}{|1-zw^*|} \leq \frac{(|z|+|w|)}{(1+|z||w|)}$ as required. **E1**

As $|z| > 1$, and $|w| > 1$, $|zw^*| > 1$, and so $1 - zw^* \neq 0$ so the division is permissible. **E1**

The working follows identically if $|z| < 1$, and $|w| < 1$ **E1 (9)**

The working for the last part (apart from final mark) may be in reverse order with $\Leftrightarrow$ signs used. If so, check carefully that the two E marks are earned and that any implication really is two way.

7. (i) $E(x) = \left(\frac{dy}{dx}\right)^2 + \frac{1}{2}\, y^4$

$\frac{dE}{dx} = 2\,\frac{dy}{dx}\,\frac{d^2y}{dx^2} + 2\,y^3\,\frac{dy}{dx}$ **M1**

$= 2\,\frac{dy}{dx}\left(\frac{d^2y}{dx^2} + y^3\right) = 0$ **A1 (**for all x, not just for $x = 0$)

Thus $E(x)$ is constant, and as $y = 1$ and $\frac{dy}{dx} = 0$, when $x = 0$, $E(x) = \frac{1}{2}$ **A1**

As $E(x) = \left(\frac{dy}{dx}\right)^2 + \frac{1}{2}\,y^4$, $\frac{1}{2}\,y^4 = E(x) - \left(\frac{dy}{dx}\right)^2 = \frac{1}{2} - \left(\frac{dy}{dx}\right)^2 \leq \frac{1}{2}$ **M1**

Thus $y^4 \leq 1$ and so $|y(x)| \leq 1$ **(*) A1 (5)**

(ii) $E(x) = \left(\frac{dv}{dx}\right)^2 + 2\,\cosh v$

$\frac{dE}{dx} = 2\,\frac{dv}{dx}\,\frac{d^2v}{dx^2} + 2\,\sinh v\,\frac{dv}{dx} = 2\,\frac{dv}{dx}\left(\frac{d^2v}{dx^2} + \sinh v\right) = 2\,\frac{dv}{dx}\left(-x\frac{dv}{dx}\right) = -2x\left(\frac{dv}{dx}\right)^2$ **M1 A1**

Thus $\frac{dE}{dx} \leq 0$ for $x \geq 0$ **(*) A1**

As $v = \ln 3$ and $\frac{dv}{dx} = 0$, when $x = 0$, $E(x) = 0 + 2\,\frac{3 + {}^1\!/_3}{2} = \frac{10}{3}$ when $x = 0$. **B1**

So as $\frac{dE}{dx} \leq 0$ for ≥ 0, $E(x) = \left(\frac{dv}{dx}\right)^2 + 2\,\cosh v \leq \frac{10}{3}$ for $x \geq 0$. **B1**

Thus $2\,\cosh v \leq \frac{10}{3} - \left(\frac{dv}{dx}\right)^2 \leq \frac{10}{3}$ for ≥ 0, and so $\cosh v(x) \leq \frac{5}{3}$ for $x \geq 0$. **(*) B1 (6)**

(iii) Let $E(x) = \left(\frac{dw}{dx}\right)^2 + 2\,(w \sinh w + \cosh w)$ **B1 + B1**

$\frac{dE}{dx} = 2\,\frac{dw}{dx}\,\frac{d^2w}{dx^2} + 2\,(w\cosh w + 2\sinh w)\,\frac{dw}{dx} = 2\,\frac{dw}{dx}\left(\frac{d^2w}{dx^2} + w\cosh w + 2\sinh w\right)$

So $\frac{dE}{dx} = 2\,\frac{dw}{dx}\left(-(5\cosh x - 4\sinh x - 3)\frac{dw}{dx}\right) = -2\left(\frac{dw}{dx}\right)^2 (5\cosh x - 4\sinh x - 3)$ **M1 A1**

$5\cosh x - 4\sinh x - 3 = 5\,\frac{e^x + e^{-x}}{2} - 4\,\frac{e^x - e^{-x}}{2} - 3 = \frac{e^{-x}}{2}\,(e^{2x} - 6e^x + 9) = \frac{e^{-x}}{2}\,(e^x - 3)^2$ **M1**

Thus $\frac{dE}{dx} \leq 0$ for $x \geq 0$ **A1**

As $w = 0$ and $\frac{dw}{dx} = \frac{1}{\sqrt{2}}$, when $x = 0$, $E(x) = \frac{1}{2} + 2 = \frac{5}{2}$ when $x = 0$. **B1**

So $\left(\frac{dw}{dx}\right)^2 + 2\,(w\sinh w + \cosh w) \leq \frac{5}{2}$ for ≥ 0.

Thus $2\cosh w \leq \frac{5}{2} - \left(\frac{dw}{dx}\right)^2 - 2w\sinh w$ for ≥ 0. **B1**

But $\left(\frac{dw}{dx}\right)^2 \geq 0$ and $w\sinh w \geq 0$ so $2\cosh w \leq \frac{5}{2}$, i.e. $\cosh w \leq \frac{5}{4}$ for $x \geq 0$. **E1 (9)**

8. $\sum_{r=0}^{n-1} e^{2i(\alpha + r\pi/n)} = e^{2i\alpha}\left(1 + e^{2i\pi/n} + e^{4i\pi/n} + \dots + e^{2i(n-1)\pi/n}\right)$ **B1**

$= e^{2i\alpha}\left(\frac{1-e^{2i\pi}}{1-e^{2i\pi/n}}\right) = 0$ as the denominator $\neq 0$

M1 A1 (GP formula) **A1** (for 0) **E1** (justification of denominator) **(5)**

$d = r\cos\theta + s$ so $s = d - r\cos\theta$ **M1 A1** (may legitimately write straight down) **(2)**

Thus $r = ks = k(d - r\cos\theta)$ **M1**

So $r = \frac{kd}{1+k\cos\theta}$ **M1 A1**

$l_j = \frac{kd}{1+k\cos\theta} + \frac{kd}{1+k\cos(\theta+\pi)}$ where $\theta = \alpha + (j-1)\,\pi/n$, $j = 1, \dots, n$ **M1**

So $l_j = \frac{kd}{1+k\cos\theta} + \frac{kd}{1-k\cos\theta} = \frac{kd(1-k\cos\theta+1+k\cos\theta)}{(1+k\cos\theta)(1-k\cos\theta)} = \frac{2kd}{1-k^2\cos^2\theta}$ **M1 A1**

B1 **(7)**

Thus $\sum_{j=1}^{n} \frac{1}{l_j} = \sum_{j=1}^{n} \frac{1-k^2\cos^2(\alpha+(j-1)\pi/n)}{2kd} = \frac{1}{2kd}\left(n - \frac{k^2}{2}\sum_{j=0}^{n-1}(\cos 2(\alpha + j\,\pi/n) + 1)\right)$

M1 **B1** **M1 A1**

$= \frac{1}{2kd}\left(n - \frac{k^2}{2}n - \frac{k^2}{2}\, Re\sum_{j=0}^{n-1} e^{2i(\alpha+j\pi/n)}\right)$ **M1**

$= \frac{(2-k^2)n}{4kd}$ as required. **(*) A1** **(6)**

9. $V = \int_x^R \pi(R^2 - t^2)dt = \pi\left[R^2 t - \frac{t^3}{3}\right]_x^R$

M1 A1 **A1**

$= \pi\left(\frac{2R^3}{3} - R^2 x + \frac{x^3}{3}\right) = \frac{\pi}{3}(2R^3 - 3R^2 x + x^3)$ **(*) A1 (4)**

$\frac{4}{3}\pi R^3 \rho_s \ddot{x} = \frac{\pi}{3}(2R^3 - 3R^2 x + x^3)\rho g - \frac{4}{3}\pi R^3 \rho_s g$ **M1** (must have all three terms) **A2** (A1 if one error)

So $4R^3 \rho_s(\ddot{x} + g) = (2R^3 - 3R^2 x + x^3)\rho g$ **(*) A1 (4)**

If $x = \frac{1}{2}R$, $\ddot{x} = 0$, **M1** so $4R^3 \rho_s = \left(2R^3 - \frac{3}{2}R^3 + \frac{R^3}{8}\right)\rho = \frac{5R^3}{8}\rho$ **A1**

and so $\rho_s = \frac{5}{32}\rho$ **A1** **(3)**

Let $x = \frac{1}{2}R + y$, **M1**

then $\frac{5}{8}R^3(\ddot{y} + g) = \left(2R^3 - 3R^2\left(\frac{1}{2}R + y\right) + \left(\frac{1}{2}R + y\right)^3\right)g$ **M1 A1**

Thus $\frac{5}{8}R^3\ddot{y} = g\left(2R^3 - \frac{3}{2}R^3 - 3R^2 y + \frac{1}{8}R^3 + \frac{3}{4}R^2 y + \frac{3}{2}Ry^2 + y^3 - \frac{5}{8}R^3\right)$ **M1 A1**

$\frac{5}{8}R^3\ddot{y} = g\left(-\frac{9}{4}R^2 y + \frac{3}{2}Ry^2 + y^3\right)$ **A1 ft but must have no constant term**

So for small y, $\ddot{y} \approx -\frac{18g}{5R}y$ **A1 ft (from previous line)**

and so the period of small oscillations $\tau = 2\pi\sqrt{\frac{5R}{18g}} = \frac{\pi}{3}\sqrt{\frac{10R}{g}}$ **M1 A1 (9)**

10. By the parallel axes rule, **M1**

the moment of inertia about P is $\frac{1}{3}Ma^2 + Mx^2 = \frac{1}{3}M(a^2+3x^2)$ **(*)**

A1 **A1** **(3)**

<u>or alternatively</u>, by the parallel axes rule, **M1** the moment of inertia about P is

$\frac{4}{3}Ma^2 - Ma^2 + Mx^2 = \frac{1}{3}M(a^2+3x^2)$ **(*)**

A1 **A1**

<u>or alternatively</u>, by integration, the moment of inertia about P is

$\int_{-a+x}^{a+x} \frac{M}{2a}u^2 du = \frac{M}{2a}\left[\frac{u^3}{3}\right]_{-a+x}^{a+x} = \frac{M}{6a}\left((a+x)^3 - (-a+x)^3\right)$ **M1A1**

$= \frac{M}{6a}(a^3 + 3a^2x + 3ax^2 + x^3 + a^3 - 3a^2x + 3ax^2 - x^3) = \frac{1}{3}M(a^2+3x^2)$ **A1**

<u>or alternatively</u>, by treating at two rods of length $a-x$ and $a+x$,

$$\frac{1}{3}\frac{a-x}{2a}M(a-x)^2 + \frac{1}{3}\frac{a-x}{2a}M(a-x)^2 = \frac{1}{3}M(a^2+3x^2)$$

M1 A1 **A1**

Conserving angular momentum about P,

$mu(a+x) = mv(a+x) + \frac{1}{3}M(a^2+3x^2)\omega$ **M1 A1 A1 A1**

where v is the velocity of the particle after impact, and ω is the angular velocity of the beam after the impact.

By Newton's experimental law of impact $(a+x)\omega - v = eu$ **M1 A1**

So substituting for v in the angular momentum equation, **M1**

$mu(a+x) = m\big((a+x)\omega - eu\big)(a+x) + \frac{1}{3}M(a^2+3x^2)\omega$ **A1**

Thus

$$mu(a+x)(1+e) = \left(m(a+x)^2 + \frac{1}{3}M(a^2+3x^2)\right)\omega$$

and so

$$\omega = \frac{3mu(a+x)(1+e)}{M(a^2+3x^2) + 3m(a+x)^2}$$

(*) A1 **(9)**

If $m = 2M$,

$$\omega = \frac{6u(a+x)(1+e)}{(a^2+3x^2)+6(a+x)^2}$$

B1

$$\frac{d\omega}{dx} = \frac{6u(1+e)}{(\,(a^2+3x^2)+6(a+x)^2)^2}\Big(\big((a^2+3x^2)+6(a+x)^2\big)-(a+x)\big(6x+12(a+x)\big)\Big)$$

M1 A1

For maximum ω , $\frac{d\omega}{dx} = 0$ **M1**

So $\big((a^2+3x^2)+6(a+x)^2\big)-(a+x)\big(6x+12(a+x)\big) = 0$

Thus $\big((a^2+3x^2)+6(a+x)^2-12(a+x)^2-6x(a+x)\big) = 0$

That is $(a^2+3x^2)-6(a+x)(a+2x) = 0$

$5a^2+18ax+9x^2 = 0 \Leftrightarrow (a+3x)(5a+3x) = 0$

So $x = -\frac{1}{3}\,a$ or $x = -\frac{5}{3}\,a$ **A1** (any correct quadratic solution method may have been used)

As $-a \le x \le a$, $x = -\frac{5}{3}\,a$ is not a feasible solution.

As

$$\frac{d\omega}{dx} = \frac{-6u(1+e)}{(\,(a^2+3x^2)+6(a+x)^2)^2}(a+3x)(5a+3x)$$

For $x < -\frac{5}{3}\,a$, $\frac{d\omega}{dx} < 0$, for $-\frac{5}{3}\,a < x < -\frac{1}{3}\,a$, $\frac{d\omega}{dx} > 0$, and for $x > -\frac{1}{3}\,a$, $\frac{d\omega}{dx} < 0$,

so the maximum ω occurs for $x = -\frac{1}{3}\,a$ **E1**

and is

$$\omega = \frac{6u\left(a+-\frac{1}{3}\,a\right)(1+e)}{\left(a^2+3\left(-\frac{1}{3}\,a\right)^2\right)+6\left(a+-\frac{1}{3}\,a\right)^2} = \frac{6u\times\frac{2}{3}\,a\times(1+e)}{\frac{4}{3}a^2+6\times\frac{4}{9}a^2} = \frac{4ua(1+e)}{4a^2} = u\,(1+e)/a$$

M1 (*) A1 **(8)**

11. The distance of the centre of the equilateral triangle from a vertex is $\frac{2}{3}\sqrt{3}a\sin\frac{\pi}{3} = a$ **B1**

So the extended length of each spring is $\frac{a}{\cos\theta}$ **M1 A1**

Thus the tension in each spring is $kmg\frac{\left(\frac{a}{\cos\theta}-a\right)}{a} = \frac{kmg(1-\cos\theta)}{\cos\theta}$ **(*) M1 A1** **(5)**

Resolving vertically $3T\sin\theta = 3mg$ so $T\sin\theta = mg$ **M1 A1**

Thus $\frac{kmg(1-\cos\theta)}{\cos\theta}\sin\theta = mg$ **M1 A 1** and so $k = \frac{\cos\theta}{(1-\cos\theta)\sin\theta}$ **B1**

If $\theta = \frac{\pi}{6}$, $k = \frac{\sqrt{3}/2}{(1-\sqrt{3}/2)1/2} = \frac{2\sqrt{3}}{2-\sqrt{3}} = \frac{2\sqrt{3}}{2-\sqrt{3}} \times \frac{2+\sqrt{3}}{2+\sqrt{3}} = 4\sqrt{3}+6$ **(*)** **M1 A1 (7)**

Taking the point of suspension as the zero level for potential energy,

when $\theta = \frac{\pi}{3}$, gravitational potential energy is $-3mga\tan\frac{\pi}{3}$

and when $\theta = \frac{\pi}{6}$, gravitational potential energy is $-3mga\tan\frac{\pi}{6}$ **B1** (both terms correct relative to chosen zero level)

When $\theta = \frac{\pi}{3}$, elastic potential energy is $\frac{3}{2}kmg\frac{\left(\frac{a}{\cos\frac{\pi}{3}}-a\right)^2}{a} = \frac{3}{2}kmga\left(\frac{1}{\cos\frac{\pi}{3}}-1\right)^2$

and when $\theta = \frac{\pi}{6}$, elastic potential energy is $\frac{3}{2}kmga\left(\frac{1}{\cos\frac{\pi}{6}}-1\right)^2$ **B1** (at least one correct or one third of these for one spring)

Therefore, conserving energy, **M1**

$-3mga\sqrt{3}+\frac{3}{2}kmga = -3mga\frac{1}{\sqrt{3}}+\frac{3}{2}kmga\left(\frac{2}{\sqrt{3}}-1\right)^2+\frac{3}{2}mV^2$ **A1** (surds) **A1** (completely correct)

So $V^2 = -2\sqrt{3}ag + (4\sqrt{3}+6)ag + \frac{2}{\sqrt{3}}ag - (4\sqrt{3}+6)\left(\frac{2}{\sqrt{3}}-1\right)^2 ag$ **M1 A1 ft**

$= ag\left(-2\sqrt{3}+4\sqrt{3}+6+\frac{2}{\sqrt{3}}-(4\sqrt{3}+6)\left(\frac{4}{3}-\frac{4}{\sqrt{3}}+1\right)\right)$

$= ag\left(-2\sqrt{3}+6+\frac{2}{\sqrt{3}}-\frac{16\sqrt{3}}{3}+16-4\sqrt{3}-8+\frac{24}{\sqrt{3}}-6\right)$

$= ag\left(8+\frac{4\sqrt{3}}{3}\right) = \frac{4ag(6+\sqrt{3})}{3}$ (*) **A1** **(8)**

12. (i) $P(X_1 = 1) = \frac{a}{n}$ **B1**

The total number of arrangements of the As and Bs is $\frac{n!}{a!b!}$ **B1**

The number of arrangements with a B in the $(k-1)$ th place and an A in the k th place is $\frac{(n-2)!}{(a-1)!(b-1)!}$ **B1**

So $P(X_k = 1) = \frac{(n-2)!}{(a-1)!(b-1)!} / \frac{n!}{a!b!} = \frac{ab}{n(n-1)}$ for $2 \le k \le n$ **B1**

$E(X_i) = 0 \times \left(1 - \frac{a}{n}\right) + 1 \times \frac{a}{n} = \frac{a}{n}$ if $i = 1$ **B1**

and $E(X_i) = 0 \times \left(1 - \frac{ab}{n(n-1)}\right) + 1 \times \frac{ab}{n(n-1)} = \frac{ab}{n(n-1)}$ if $i \neq 1$ **B1**

$E(S) = E(\sum_{i=1}^{n} X_i) = \frac{a}{n} + (n-1)\frac{ab}{n(n-1)} = \frac{a}{n} + \frac{ab}{n} = \frac{a(b+1)}{n}$ **(*) B1** **(7)**

(ii) a) $X_1 X_j = 1$ only if the first letter is an A, the $(j-1)$ th letter is a B, and the j th letter is an A. **E1**

This has probability $\frac{(n-3)!}{(a-2)!(b-1)!} / \frac{n!}{a!b!} = \frac{a(a-1)b}{n(n-1)(n-2)}$ **B1**

So $E(X_1 X_j) = 0 \times \left(1 - \frac{a(a-1)b}{n(n-1)(n-2)}\right) + 1 \times \frac{a(a-1)b}{n(n-1)(n-2)} = \frac{a(a-1)b}{n(n-1)(n-2)}$ **(*) B1** **(3)**

b) $X_i X_j = 1$ only if the $(i-1)$ th letter is a B, and the i th letter is an A, the $(j-1)$ th letter is a B, and the j th letter is an A. **E1**

This has probability $\frac{(n-4)!}{(a-2)!(b-2)!} / \frac{n!}{a!b!} = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$ **B1**

So $E(X_i X_j) = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$, and thus $\sum_{j=i+2}^{n} E(X_i X_j) = (n-i-1)\frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}$ **B1**

and so $\sum_{i=2}^{n-2}\left(\sum_{j=i+2}^{n} E(X_i X_j)\right) = \sum_{i=2}^{n-2}\left((n-i-1)\frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)}\right)$ **B1**

$= \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)} \sum_{i=2}^{n-2}(n-i-1) = \frac{a(a-1)b(b-1)}{n(n-1)(n-2)(n-3)} \frac{(n-3)(n-2)}{2}$

$= \frac{a(a-1)b(b-1)}{2n(n-1)}$ **(*) B1** **(5)**

c) $S^2 = \sum_{i=1}^{n} {X_i}^2 + \sum_{i=1}^{n-1} \sum_{j=i+1}^{n} 2X_i X_j$ **B1**

So $E(S^2) = \frac{a}{n} + (n-1)\frac{ab}{n(n-1)} + 2(n-2)\frac{a(a-1)b}{n(n-1)(n-2)} + 2\frac{a(a-1)b(b-1)}{2n(n-1)}$

$$= \frac{a(b+1)}{n} + \frac{2a(a-1)b + a(a-1)b(b-1)}{n(n-1)} = \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)}$$ **B1**

Thus $Var(S) = \frac{a(b+1)}{n} + \frac{a(a-1)b(b+1)}{n(n-1)} - \frac{a^2(b+1)^2}{n^2} = \frac{a(b+1)\left(n-a(b+1)\right)}{n^2} + \frac{a(a-1)b(b+1)}{n(n-1)}$ **M1 A1**

$$= \frac{a(b+1)(a+b-ab-a)}{n^2} + \frac{a(a-1)b(b+1)}{n(n-1)} = \frac{a(b+1)b(1-a)}{n^2} + \frac{a(a-1)b(b+1)}{n(n-1)}$$

$$= \frac{a(a-1)b(b+1)\left(n-(n-1)\right)}{n^2(n-1)} = \frac{a(a-1)b(b+1)}{n^2(n-1)}$$ **(*) A1** **(5)**

Many of the marks can be implied by later correct expressions, but beware ‘reasoned methods’ that arise from working round the given answers.

13. a) (i) $0 \le f(x) \le M$ and so

$\int_0^x 0\,dt \le \int_0^x f(t)\,dt \le \int_0^x M\,dt$ **M1**

Thus $0 \le [F(t)]_0^x \le [Mt]_0^x$ **M1**

and so $0 \le F(x) - F(0) \le Mx$, that is $0 \le F(x) \le Mx$ **(*) A1 (3)**

(ii)

$$\int_0^1 2g(x)\,F(x)f(x)dx = \left[g(x)\big(F(x)\big)^2\right]_0^1 - \int_0^1 g'(x)\,\big(F(x)\big)^2 dx$$

integrating by parts $u = g(x)$, $u' = g'(x)$, $v' = 2F(x)f(x)$, $v = \big(F(x)\big)^2$ **M1 A1**

But $\left[g(x)\big(F(x)\big)^2\right]_0^1 = g(1)\big(F(1)\big)^2 - g(0)\big(F(0)\big)^2 = g(1) - 0 = g(1)$ **M1 A1**

So

$$\int_0^1 2g(x)\,F(x)f(x)dx = g(1) - \int_0^1 g'(x)\,\big(F(x)\big)^2 dx$$

which is the required result. **(*) A1 (5)**

b) (i) As $kF(y)f(y)$ is a probability density function,

$\int_0^1 k\,F(y)f(y)dy = 1$ **M1**

Using the result of a) (ii) with $g(x) = \frac{1}{2}k$, $\frac{1}{2}k = 1$ so $k = 2$ **M1 (*) A1 (3)**

(Note that $g(x) = \lambda k$ for any choice of λ could be used by candidates)

(ii) $E(Y^n) = \int_0^1 y^n\,2F(y)f(y)dy \le \int_0^1 y^n\,2My f(y)dy = 2M\int_0^1 y^{n+1}\,f(y)dy = 2M\mu_{n+1}$

M1 **(*) A1**

Using a) (ii), $E(Y^n) = \int_0^1 y^n\,2F(y)f(y)dy = \frac{1}{2}\times 2\times 1^n - \frac{1}{2}\int_0^1 2ny^{n-1}\big(F(y)\big)^2 dy$

$= 1 - n\int_0^1 y^{n-1}\big(F(y)\big)^2 dy$ **M1**

$\int_0^1 y^{n-1}\big(F(y)\big)^2 dy \le \int_0^1 y^{n-1}My\,F(y)dy = M\int_0^1 y^n\,F(y)dy$ **M1**

Integrating by parts $u = F(y)$, $u' = f(y)$, $v' = y^n$, $v = \frac{y^{n+1}}{n+1}$

$\int_0^1 y^n\,F(y)dy = \left[F(y)\frac{y^{n+1}}{n+1}\right]_0^1 - \int_0^1 \frac{y^{n+1}}{n+1}f(y)dy = \frac{1}{n+1} - \frac{1}{n+1}\mu_{n+1}$ **M1**

So

$$E(Y^n) \geq 1 - nM\left(\frac{1}{n+1} - \frac{1}{n+1}\mu_{n+1}\right)$$

That is

$E(Y^n) \geq 1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1}$ **(*) A1** **(6)**

Thus

$$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \leq E(Y^n) \leq 2M\mu_{n+1}$$

(iii) Hence

$$1 + \frac{nM}{n+1}\mu_{n+1} - \frac{nM}{n+1} \leq 2M\mu_{n+1}$$

M1

So

$$[2M(n+1) - nM]\mu_{n+1} \geq (n+1) - nM$$

$$M(n+2)\mu_{n+1} \geq (n+1) - nM$$

$$\mu_{n+1} \geq \frac{(n+1)}{(n+2)M} - \frac{n}{(n+2)}$$

A1

and hence

$$\mu_n \geq \frac{n}{(n+1)M} - \frac{n-1}{n+1}$$

(*) A1 **(3)**