1. $$(1+ax)(1+bx)(1+cx) = (1+(a+b)x+x^2)(1+cx)$$

$= 1+(a+b+c)x+(bc+ca+ab)x^2+abcx^3 = 1+qx^2+rx^3$ **M1 expanding fully**

so equating coefficients , $a+b+c=0$ as stated, $bc+ca+ab=q$, and $abc=r$ **(*) E1 (2)**

(i)

$$\ln(1+qx^2+rx^3) = \ln\big((1+ax)(1+bx)(1+cx)\big) = \ln(1+ax)+\ln(1+bx)+\ln(1+cx)$$

M1 use stem **M1 ln manipulation**

$$= (ax) - \frac{(ax)^2}{2} + \frac{(ax)^3}{3} - \cdots + (-1)^{n+1}\frac{(ax)^n}{n} + \cdots + similar\ expressions\ in\ b\ \&\ c$$

M1 A1 ln series

Thus the coefficient of x^n is $(-1)^{n+1}\frac{a^n+b^n+c^n}{n}$ i.e. $(-1)^{n+1}S_n$ where $S_n = \frac{a^n+b^n+c^n}{n}$ **(*) A1(5)**

(ii) $\ln(1+qx^2+rx^3) = (qx^2+rx^3) - \frac{(qx^2+rx^3)^2}{2} + \frac{(qx^2+rx^3)^3}{3} - \cdots$

M1 ln series

$= qx^2+rx^3 - \frac{q^2x^4}{2} - \frac{2qrx^5}{2} - \frac{r^2x^6}{2} + \frac{q^3x^6}{3} + \frac{3q^2rx^7}{3} + \frac{3qr^2x^8}{3} + \frac{r^3x^9}{3} - \frac{q^4x^8}{4} - \frac{4q^3rx^9}{4} - \cdots$

M1 expansion

Equating coefficients with those from (i) , **M1**

$-S_2 = q$, $S_3 = r$, and $S_5 = -qr$, and hence $S_2S_3 = -qr = S_5$ as required.

A1 **(*) A1 (5)**

(iii) $S_7 = q^2r$ and $S_2S_5 = -q \times -qr = q^2r = S_7$ as required.

M1 **M1** **(*) A1 (3)**

(iv) $S_2S_7 = -q \times q^2r = -q^3r$, $S_9 = \frac{r^3}{3} - q^3r$, $S_2S_7 \neq S_9$ provided $\frac{r^3}{3} \neq 0$, i.e. $r \neq 0$

B1 **B1** **B1**

e.g. $a=2,\ b=-1,\ c=-1$, $a+b+c=0$, $abc=r=2\neq 0$ **B1 any correct example**

E1 justified (5)

2 (i) Let $u = \cosh x$, then $\frac{du}{dx} = \sinh x$ **B1 (may not be explicitly stated)**

so $\int \frac{\sinh x}{\cosh 2x}\, dx = \int \frac{\sinh x}{2\cosh^2 x - 1}\, dx = \int \frac{1}{2u^2-1}\, du = \int \frac{1/2}{\sqrt{2}\,u-1} - \frac{1/2}{\sqrt{2}\,u+1}\, du$

M1 'double angle' M1 complete change of variable M1 partial fractions

$= \frac{1}{2\sqrt{2}} \ln\left|\sqrt{2}\,u - 1\right| - \frac{1}{2\sqrt{2}} \ln\left|\sqrt{2}\,u + 1\right| + C$

$= \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}\,u-1}{\sqrt{2}\,u+1}\right| + C = \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}\,\cosh x-1}{\sqrt{2}\,\cosh x+1}\right| + C$ as required.

M1 integration and ln manipulation A1 (*) A1 cso (7)

SC3 instead of M1 partial fractions, and next M1A1 if formula book used for integral

(ii) Let $u = \sinh x$, then $\frac{du}{dx} = \cosh x$ **M1 A1**

so $\int \frac{\cosh x}{\cosh 2x}\, dx = \int \frac{\cosh x}{1+2\sinh^2 x}\, dx = \int \frac{1}{1+2u^2}\, du = \int \frac{1/2}{1/2 + u^2}\, du = \frac{\sqrt{2}}{2} \tan^{-1}\sqrt{2}\,u + c$

M1 M1 M1

$= \frac{\sqrt{2}}{2} \tan^{-1}\left(\sqrt{2}\,\sinh x\right) + c$ **A1 (6)**

(iii) Let $u = e^x$, then $\frac{du}{dx} = e^x$ **M1**

so $\int_0^1 \frac{1}{1+u^4}\, du = \int_{-\infty}^0 \frac{e^x}{1+e^{4x}}\, dx = \int_{-\infty}^0 \frac{e^{-x}}{e^{-2x}+e^{2x}}\, dx = \frac{1}{2}\int_{-\infty}^0 \frac{\cosh x - \sinh x}{\cosh 2x}\, dx$

M1 A1 M1 M1

$= \frac{1}{2}\left[\frac{\sqrt{2}}{2} \tan^{-1}\left(\sqrt{2}\,\sinh x\right) - \frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}\,\cosh x-1}{\sqrt{2}\,\cosh x+1}\right|\right]_{-\infty}^0 = \frac{1}{2}\left\{-\frac{1}{2\sqrt{2}} \ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right| - \frac{\sqrt{2}}{2}\left(\frac{-\pi}{2}\right)\right\}$

M1

$= \frac{\sqrt{2}}{8}\left(\pi + 2\ln\left(\sqrt{2}+1\right)\right)$ **(*) A1 (7)**

Alternatively,

Let $u = e^{-x}$, then $\frac{du}{dx} = -e^{-x}$ **M1**

so $\int_0^1 \frac{1}{1+u^4}\, du = \int_{\infty}^0 \frac{-e^{-x}}{1+e^{-4x}}\, dx = \int_0^{\infty} \frac{e^x}{e^{-2x}+e^{2x}}\, dx = \frac{1}{2}\int_0^{\infty} \frac{\cosh x + \sinh x}{\cosh 2x}\, dx$

M1 A1 M1 M1

$$= \frac{1}{2}\left[\frac{\sqrt{2}}{2}\tan^{-1}\left(\sqrt{2}\,\sinh x\right) + \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}\,\cosh x - 1}{\sqrt{2}\,\cosh x + 1}\right|\right]_0^{\infty} = \frac{1}{2}\left\{\frac{\sqrt{2}}{2}\left(\frac{-\pi}{2}\right) - \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{2}-1}{\sqrt{2}+1}\right|\right\}$$

M1

$$= \frac{\sqrt{2}}{8}\left(\pi + 2\ln\left(\sqrt{2}+1\right)\right)$$ **(*) A1 (7)**

3. Shortest distance between $y = mx + c$ and $y^2 = 4ax$ if they do not intersect is distance between point on parabola where the tangent has gradient m **M1**

and the point where the normal at that point intersects $y = mx + c$.

$(at^2, 2at)$ is a general point on $y^2 = 4ax$, and $\frac{dy}{dx} = \frac{1}{t}$, so $\frac{1}{t} = m$ **M1**

thus the distance required is that between $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ **A1**

(alternatively $y^2 = 4ax$, $2y\frac{dy}{dx} = 4a$, so $\frac{dy}{dx} = \frac{2a}{y} = m$ **M1** to give $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$ **A1**)

and the intersection of $y = mx + c$ and $y - \frac{2a}{m} = \frac{-1}{m}\left(x - \frac{a}{m^2}\right)$.

Solving for the intersection $mx + c = \frac{-1}{m}\left(x - \frac{a}{m^2}\right) + \frac{2a}{m}$

$(m^2 + 1)m^2x = a(1 + 2m^2) - m^3c$ so $x = \frac{a(1+2m^2)-m^3c}{(m^2+1)m^2}$ and

$y = \frac{a(1+2m^2)-m^3c}{(m^2+1)m} + c = \frac{a(1+2m^2)+mc}{(m^2+1)m}$ **M1**

Thus the shortest distance squared is

$\left(\frac{a(1+2m^2)-m^3c}{(m^2+1)m^2} - \frac{a}{m^2}\right)^2 + \left(\frac{a(1+2m^2)+mc}{(m^2+1)m} - \frac{2a}{m}\right)^2$

$= \frac{1}{(m^2+1)^2m^4}\left((m^2a - m^3c)^2 + m^2(-a + mc)^2\right)$

$= \frac{1}{(m^2+1)^2m^4}(mc - a)^2(m^4 + m^2) = \frac{(mc-a)^2}{(m^2+1)m^2}$ **M1**

and thus the shortest distance is $\frac{(mc-a)}{m\sqrt{(m^2+1)}}$ as required. **(*) A1**

(alternatively, using the perpendicular distance formula

$$\frac{\left|m \times \frac{a}{m^2} - \frac{2a}{m} + c\right|}{\sqrt{(m^2+1)}} = \frac{\left|-\frac{a}{m} + c\right|}{\sqrt{(m^2+1)}} = \frac{(mc - a)}{m\sqrt{(m^2+1)}}$$

M1 **M1** **(*) A1**)

The condition that they do not meet is that solving $y = mx + c$ and $y^2 = 4ax$ simultaneously has no real roots.

i.e. $(mx + c)^2 = 4ax$ has no real roots,

$m^2x^2 + 2x(mc - 2a) + c^2 = 0$ has no real roots,

in other words the discriminant is negative, $(mc - 2a)^2 - m^2c^2 < 0$ **M1**

$4a^2 - 4mca < 0$, $4a(a - mc) < 0$

so if $a > 0$, $mc > a$ **E1**

If $mc \leq a$, the curve and line meet/intersect and so the shortest distance is zero. **B1 (9)**

(ii) The shortest distance between $(p, 0)$ and $y^2 = 4ax$ is either p if the closest point on $y^2 = 4ax$ to $(p, 0)$ is $(0,0)$, or is the distance along the normal that passes through $(p, 0)$ to the point where it is the normal. **M1**

The normal at $(at^2, 2at)$ is $y - 2at = -t\,(x - at^2)$ **M1**

and if this passes through $(p, 0)$,

$-2at = -t\,(p - at^2)$ so $p = 2a + at^2$ **A1**

Thus if $p - 2a < 0$, the only normal passing through $(p, 0)$ is $y = 0$ **M1** and so the distance is p, i.e. if $\frac{p}{a} < 2$. **A1**

If $\frac{p}{a} \geq 2$, the distance squared is $(2a)^2 + \left(2a\sqrt{\frac{p}{a} - 2}\right)^2 = 4a^2 + 4ap - 8a^2 = 4ap - 4a^2$

Thus the distance is $2\sqrt{a(p-a)}$. **M1 A1** **(7)**

That is $\frac{p}{a} < 2$, distance is p

$\frac{p}{a} \geq 2$ **,** distance is $2\sqrt{a(p-a)}$

So for the circle, if $\frac{p}{a} < 2$, the distance will be $p - b$ if $p > b$, **B1** and 0 otherwise **B1** , and if $\frac{p}{a} \geq 2$, the distance will be $2\sqrt{a(p-a)} - b$ if $4a(p-a) > b^2$ **B1** or 0 otherwise. **B1** (4)

That is $\frac{p}{a} < 2$, if $p > b$ distance is p

otherwise distance is 0

$\frac{p}{a} \geq 2$ **,** distance is $2\sqrt{a(p-a)} - b$ if $4a(p-a) > b^2$

otherwise distance is 0

4. (i) $I_1 = \int_0^1 (y' + y\tan x)^2\, dx$

$= \int_0^1 (y')^2 + 2yy'\tan x + y^2\tan^2 x\;\, dx$

$= \int_0^1 (y')^2 + 2yy'\tan x + y^2(\sec^2 x - 1)\, dx$ **M1**

$= \int_0^1 (y')^2 - y^2\, dx \;+\; \int_0^1 2yy'\tan x + y^2\sec^2 x\;\, dx$ **A1**

$= I + [y^2\tan x]_0^1 = I + y^2(1)\tan 1 - y^2(0)\tan 0 = I + 0 - 0 = I$ as required.

A1 **(*) A1**

$(y' + y\tan x)^2 \geq 0 \;\Rightarrow\; I_1 \geq 0 \;\Rightarrow I \geq 0$ **E1**

$I = 0$ if and only if $y' + y\tan x = 0 \qquad \forall x\,,\; 0 \leq x \;\leq 1$

$y' = -y\tan x \qquad \forall x\,,\; 0 \leq x \;\leq 1$

$\frac{y'}{y} = -\tan x \qquad \forall x\,,\; 0 \leq x \;\leq 1$ **M1**

$\ln y = \ln(\cos x) + c \qquad \forall x\,,\; 0 \leq x \;\leq 1$ **A1**

$y = A\cos x \qquad \forall x\,,\; 0 \leq x \;\leq 1$

$y = 0\,, x = 1 \;\Rightarrow A = 0$ so $y = 0 \qquad \forall x\,,\; 0 \leq x \;\leq 1$ **M1 A1 (9)**

(ii) $\int_0^1 (y' + ay\tan bx)^2 dx = \int_0^1 (y')^2 + 2ay'y\tan bx + a^2y^2\tan^2 bx\;\, dx$

$= \int_0^1 (y')^2 + 2ay'y\tan bx + a^2y^2\sec^2 bx - a^2y^2\, dx$ **M1**

$= \int_0^1 (y')^2 - a^2y^2\, dx \;+\; \int_0^1 2ayy'\tan bx + a^2y^2\sec^2 bx\;\, dx$ **A1**

$= J + \;[ay^2\tan bx]_0^1$ **A1** provided $b = a$ **B1**

$[ay^2\tan bx]_0^1 = \; ay^2(1)\tan b - \; ay^2(0)\tan b0 = 0 - 0 = 0$

So $J = \int_0^1 (y' + ay\tan bx)^2 dx$ **A1**

$(y' + ay\tan bx)^2 \geq 0$ and so $J \geq 0$ **A1**

The argument requires no discontinuity in the interval so $a = b < \frac{\pi}{2}$**.** **B1 (7)**

If $a = \frac{\pi}{2}$, let $y = \cos ax$, B**1** then $y' = -a\sin ax$ so $(y')^2 - a^2y^2 = a^2\sin^2 ax - a^2\cos^2 ax$,
and $\int_0^1 -a^2\cos 2ax\, dx = \;[-a\sin 2ax]_0^1 = -a\sin\pi + a\sin 0 = 0$, and $x = 1$, $\cos a = 0$ but y is

M1 **A1**

not identically zero. **E1 (4)**

5. ABCD is a parallelogram if and only if $\overrightarrow{AB} = \overrightarrow{DC}$, **B1**

i.e. if and only if $b - a = c - d$ **M1**

which rearranged gives $a + c = b + d$ as required. **(*) A1** **(3)**

Further, to be a square as well, angle $ABC = 90^o$, and $|AB| = |BC|$. **M1**

Thus $|c - b| = |b - a|$ and $(c - b) = arg(b - a) + 90^o$, so $c - b = i(b - a)$. **B1**

So $a + c = b + d$ and $ia + c = (i + 1)b$, and thus $a(1 - i) = d - ib$ yielding

$a(1 - i)(1 + i) = (d - ib)(1 + i)$, hence $a = \frac{1}{2}\big((1 - i)b + (1 + i)d\big)$ and so

$c = b + d - \frac{1}{2}\big((1 - i)b + (1 + i)d\big) = \frac{1}{2}\big((1 + i)b + (1 - i)d\big)$

So $i(a - c) = i\left[\frac{1}{2}\big((1 - i)b + (1 + i)d\big) - \frac{1}{2}\big((1 + i)b + (1 - i)d\big)\right] = i[-ib + id] = b - d$ **B1 (3)**

(alternatively, ABCD square $\Leftrightarrow$ ABCD parallelogram **B1** & diagonals equal length and perpendicular **B1** $\Leftrightarrow$ $i(a - c) = b - d$ **B1 (3)**)

(i) angle $PXQ = 90^o$, and $|PX| = |XQ|$ so, replicating result with ABC in stem, M1

$q - x = i(x - p)$ **A1** so $(1 + i)x = ip + q$, $(1 - i)(1 + i)x = (1 - i)(ip + q\ )$, **M1** and hence $2x = (1 + i)p + (1 - i)q$ and $x = \frac{1}{2}\big(p(1 + i) + q(1 - i)\big)$ **A1** **(4)**

(ii) From the stem, XYZT is a square if and only if $i(x - z) = y - t$, and $x + z = y + t$ **B1 B1**

$\Leftrightarrow i\left(\frac{1}{2}\big(p(1 + i) + q(1 - i)\big) - \frac{1}{2}\big(r(1 + i) + s(1 - i)\big)\right) = \frac{1}{2}\big(q(1 + i) + r(1 - i)\big) - \frac{1}{2}\big(s(1 + i) + p(1 - i)\big)$ M1 A1 A1

$\Leftrightarrow p(i - 1) + q(1 + i) + r(1 - i) - s(1 + i) = p(i - 1)i + q(1 + i) + r(1 - i) - s(1 + i)$ which is trivially true **M1A1** **(7)**

and

$\frac{1}{2}\big(p(1 + i) + q(1 - i)\big) + \frac{1}{2}\big(r(1 + i) + s(1 - i)\big) = \frac{1}{2}\big(q(1 + i) + r(1 - i)\big) + \frac{1}{2}\big(s(1 + i) + p(1 - i)\big)$ **M1 A1**

$\Leftrightarrow ip - iq + ir - is = 0 \Leftrightarrow p + r = q + s$ so PQRS is a parallelogram. **B1** **(3)**

6. $f''(t) > 0$ for $0 < t < x_0 \implies \int_0^{t_0} f''(t)\,dt > 0$ where $0 < t_0 < x_0$ **B1**

So $[f'(t)]_0^{t_0} > 0$ and thus $f'(t_0) - f'(0) > 0$ i.e. $f'(t_0) > 0$

M1 **A1** **A1**

Repeating the same argument for $f'(t)$ thus gives $f(t) > 0$ **E1 (5)**

(i) Let $f(x) = 1 - \cos x \cosh x$ **M1**

then $f'(x) = \sin x \cosh x - \cos x \sinh x$ **M1 A1**

and $f''(x) = \cos x \cosh x + \sin x \sinh x - \cos x \cosh x + \sin x \sinh x = 2 \sin x \sinh x$ **A1**

$f(0) = 0$ and $f'(0) = 0$

For $0 < x < \pi$, $\sin x > 0$ and $\sinh x > 0$ so $f''(x) > 0$ and so this is true for $0 < x < \pi/2$ in particular. **B1**

Hence by the stem, $1 - \cos x \cosh x > 0$ for $0 < x < \pi/2$ i.e. $\cos x \cosh x < 1$ as required.

(*) A1 (6)

(ii) Let $g(x) = x^2 - \sin x \sinh x$ **M1**

then $g'(x) = 2x - \cos x \sinh x - \sin x \cosh x$ **A1**

and $g''(x) = 2 + \sin x \sinh x - \cos x \cosh x - \cos x \cosh x - \sin x \sinh x = 2 - 2\cos x \cosh x$

A1

Thus $g''(x) = 2f(x)$ where $f(x)$ is as defined in part (i).

$g(0) = 0$ and $g'(0) = 0$ and from part (i) $g''(x) > 0$ for $0 < x < \pi/2$, so $x^2 - \sin x \sinh x > 0$ for $0 < x < \pi/2$. **E1**

As $x^2 > \sin x \sinh x$ and for $0 < x < \pi/2$, as $x > 0$, $\sin x > 0$, $\sinh x > 0$

then $\frac{x}{\sinh x} > \frac{\sin x}{x}$ **E1 (5)**

Let $h(x) = \sin x \cosh x - x$ **M1**

then $h'(x) = \cos x \cosh x + \sin x \sinh x - 1$

and $h''(x) = -\sin x \cosh x + \cos x \sinh x + \cos x \sinh x + \sin x \cosh x = 2 \cos x \sinh x$ **A1**

$h(0) = 0$ and $h'(0) = 0$ and $h''(x) > 0$ for $0 < x < \pi/2$, so $\sin x \cosh x - x > 0$ for $0 < x < \pi/2$. **E1**

As $\sin x \cosh x > x$ and for $0 < x < \pi/2$, as $x > 0$, $\sin x > 0, \cosh x > 0$

then $\frac{\sin x}{x} > \frac{1}{\cosh x}$ **A1 (4)**

7. (i) $\widehat{P_1QP_4} = \widehat{P_2QP_3}$ vertically opposite angles **E1**

$\widehat{P_1P_4Q} = \widehat{P_1P_4P_2}$ same angle, $= \widehat{P_1P_3}P_2$ angles subtended by chord P_1P_2, $= \widehat{QP_3}P_2$ same angle **E1**

Thus ΔP_1QP_4 is similar to ΔP_2QP_3 **E1** as two angles, and hence three, are equal. **E1**

So $\frac{P_1Q}{QP_4} = \frac{P_2Q}{QP_3}$ and therefore, $P_1Q \cdot QP_3 = P_2Q \cdot QP_4$ as required. (*) **B1 cso (5)**

(ii) If $\boldsymbol{q}$ is the position vector of the point Q, as Q lies on P_1P_3, $\boldsymbol{q} = \boldsymbol{p_1} + \lambda(\boldsymbol{p_3} - \boldsymbol{p_1})$ **M1**

where $\lambda \neq 0, 1$. So $\boldsymbol{q} = (1-\lambda)\boldsymbol{p_1} + \lambda\boldsymbol{p_3}$ **A1**

Similarly, as Q lies on P_2P_4, $\boldsymbol{q} = (1-\mu)\boldsymbol{p_2} + \mu\boldsymbol{p_4}$ where $\mu \neq 0, 1$. **B1**

Equating these expressions we have $(1-\lambda)\boldsymbol{p_1} + \lambda\boldsymbol{p_3} = (1-\mu)\boldsymbol{p_2} + \mu\boldsymbol{p_4}$ and re-arranging, **M1**

$(1-\lambda)\boldsymbol{p_1} - (1-\mu)\boldsymbol{p_2} + \lambda\boldsymbol{p_3} - \mu\boldsymbol{p_4} = \boldsymbol{0}$, **A1**

and thus with $a_1 = 1-\lambda$, $a_2 = -(1-\mu)$, $a_3 = \lambda$, and $a_3 = -\mu$, none of which is zero, $\sum_{i=1}^{4} a_i = 0$ **A1** and $\sum_{i=1}^{4} a_i\boldsymbol{p_i} = \boldsymbol{0}$ **(6)**

(iii) If $a_1 + a_3 = 0$, then $a_2 + a_4 = 0$, and so $a_1\boldsymbol{p_1} + a_2\boldsymbol{p_2} - a_1\boldsymbol{p_3} - a_2\boldsymbol{p_4} = \boldsymbol{0}$ **E1**

which means that $a_1(\boldsymbol{p_1} - \boldsymbol{p_3}) = a_2(\boldsymbol{p_4} - \boldsymbol{p_2})$ **. B1** P_1P_3 and P_2P_4 are non-zero as the four points are distinct, and are non-parallel as they intersect at Q. **E1** Thus $a_1 = a_2 = 0$ and hence $a_1 = a_2 = a_3 = a_4 = 0$ which contradicts (*). **E1 (4)**

$\frac{a_1\boldsymbol{p_1} + a_3\boldsymbol{p_3}}{a_1+a_3} = \boldsymbol{p_1} + \frac{a_3(\boldsymbol{p_3} - \boldsymbol{p_1})}{a_1+a_3}$ and so the point with this position vector lies on P_1P_3. **E1**

Also $\frac{a_1\boldsymbol{p_1} + a_3\boldsymbol{p_3}}{a_1+a_3} = \frac{-(a_2\boldsymbol{p_2} + a_4\boldsymbol{p_4})}{-(a_2+a_4)} = \frac{a_2\boldsymbol{p_2} + a_4\boldsymbol{p_4}}{a_2+a_4}$ and so, by the same argument, the point with this position vector lies on P_2P_4 and hence is the point of intersection Q. **E1 (2)**

As $P_1Q \cdot QP_3 = P_2Q \cdot QP_4$, $\frac{a_3}{a_1+a_3}P_1P_3 \cdot \frac{a_1}{a_1+a_3}P_1P_3 = \frac{a_4}{a_2+a_4}P_2P_4 \cdot \frac{a_2}{a_2+a_4}P_2P_4$ M1

That is $\frac{a_1\, a_3}{(a_1+a_3)^2}(P_1P_3)^2 = \frac{a_2\, a_4}{(a_2+a_4)^2}(P_2P_4)^2$ **A1**

and as $(a_1 + a_3) + (a_2 + a_4) = 0$, $a_1\, a_3(P_1P_3)^2 = a_2\, a_4(P_2P_4)^2$ **(*) A1 (3)**

8.

$$\sum_{r=k^n}^{k^{n+1}-1} f(r) = f(k^n) + f(k^n+1) + \cdots + f(k^{n+1}-1)$$

$f(k^n) > f(k^n+1) > \cdots > f(k^{n+1}-1) > f(k^{n+1})$ **M1**

Thus

$[(k^{n+1}-1)-(k^n-1)]f(k^{n+1}) < \sum_{r=k^n}^{k^{n+1}-1} f(r) < [(k^{n+1}-1)-(k^n-1)]f(k^n)$ **M1**

$[(k^{n+1}-1)-(k^n-1)]f(k^{n+1}) = (k^{n+1}-k^n)f(k^{n+1}) = k^n(k-1)f(k^{n+1})$ **M1**

and similarly $[(k^{n+1}-1)-(k^n-1)]f(k^n) = k^n(k-1)f(k^n)$ so

$k^n(k-1)f(k^{n+1}) < \sum_{r=k^n}^{k^{n+1}-1} f(r) < k^n(k-1)f(k^n)$ **(*) A1 (4)**

as required.

(i) Applying the result of the stem with $f(r) = {}^1/_r$, $k = 2$, **B1** and

$$\sum_{r=1}^{2^{N+1}-1} {}^1/_r = \sum_{r=2^0}^{2^1-1} {}^1/_r + \sum_{r=2^1}^{2^2-1} {}^1/_r + \ldots + \sum_{r=2^N}^{2^{N+1}-1} {}^1/_r$$

M1

so

$$\sum_{r=1}^{2^{N+1}-1} {}^1/_r < 2^0(2-1)\left({}^1/_{2^0}\right) + 2^1(2-1)\left({}^1/_{2^1}\right) + \cdots + 2^N(2-1)\left({}^1/_{2^N}\right)$$

M1

and

$$\sum_{r=1}^{2^{N+1}-1} {}^1/_r > 2^0(2-1)\left({}^1/_{2^1}\right) + 2^1(2-1)\left({}^1/_{2^2}\right) + \cdots + 2^N(2-1)\left({}^1/_{2^{N+1}}\right)$$

M1

i.e.

$$\frac{N+1}{2} < \sum_{r=1}^{2^{N+1}-1} {}^1/_r < N+1$$

(*) A1 (5)

$$\sum_{r=1}^{\infty} {}^{1}\!/_{r} > \frac{N+1}{2}$$

for any N and hence the sum $\sum_{r=1}^{\infty} {}^{1}\!/_{r}$ does not converge. **E1 (1)**

(ii) Applying the result of the stem with $f(r) = {}^{1}\!/_{r^3}$, $k = 2$, and

$$\sum_{r=1}^{2^{N+1}-1} {}^{1}\!/_{r^3} = \sum_{r=2^0}^{2^1-1} {}^{1}\!/_{r^3} + \sum_{r=2^1}^{2^2-1} {}^{1}\!/_{r^3} + \dots + \sum_{r=2^N}^{2^{N+1}-1} {}^{1}\!/_{r^3}$$

M1

$$\sum_{r=1}^{2^{N+1}-1} {}^{1}\!/_{r^3} < 2^0(2-1)\left({}^{1}\!/_{2^0}\right) + 2^1(2-1)\left({}^{1}\!/_{2^3}\right) + \dots + 2^N(2-1)\left({}^{1}\!/_{2^{3N}}\right)$$

M1

$$= 1 + {}^{1}\!/_{2^2} + \dots + {}^{1}\!/_{2^{2N}} = \frac{1 - {}^{1}\!/_{2^{2N+2}}}{1 - {}^{1}\!/_{2^2}}$$

A1

Taking the limit as $N \to \infty$ $\sum_{r=1}^{\infty} {}^{1}\!/_{r^3} \leq \frac{1}{1-{}^{1}\!/_{2^2}} = \frac{4}{3} = 1\frac{1}{3}$ **(*) A1 (4)**

(iii) $S(1000)$ is the set of 3 digit numbers in which each digit can be 0,1,3,4, … ,9 excluding 000, so it has $9^3 - 1$ elements. **E1 (1)**

If $f(r) = {}^{1}\!/_{r}$ for integer r unless r has one or more 2 s in its decimal representation in which case $f(r) = 0$, then **M1**

$$\sum_{r=10^0}^{10^1-1} f(r) < (9-1) \times 1$$

A1

$$\sum_{r=10^1}^{10^2-1} f(r) < (9^2 - 9) \times \frac{1}{10} = 9(9-1) \times \frac{1}{10}$$

$$\sum_{r=10^2}^{10^3-1} f(r) < (9^3 - 9^2) \times \frac{1}{10^2} = 9^2(9-1) \times \frac{1}{10^2}$$

and so on

so $\sigma(n) < 8\left(1 + \frac{9}{10} + \left(\frac{9}{10}\right)^2 + \cdots\right) = \frac{8}{1-\frac{9}{10}} = 80$ as required.

A1 **M1 (*) A1 (5)**

9.

$$\mathbf{r} = \frac{\mathrm{kt} - 1 + \mathrm{e}^{-\mathrm{kt}}}{\mathrm{k}^2}\mathbf{g} + \frac{1 - \mathrm{e}^{-\mathrm{kt}}}{\mathrm{k}}\mathbf{u}$$

$\mathbf{v} = \frac{\mathrm{d}\mathbf{r}}{\mathrm{dt}} = \frac{\mathrm{k} - \mathrm{ke}^{-\mathrm{kt}}}{\mathrm{k}^2}\mathbf{g} + \frac{\mathrm{ke}^{-\mathrm{kt}}}{\mathrm{k}}\mathbf{u} = \frac{1 - \mathrm{e}^{-\mathrm{kt}}}{\mathrm{k}}\mathbf{g} + \mathrm{e}^{-\mathrm{kt}}\mathbf{u}$ **M1 A1 (2)**

$\mathbf{a} = \frac{\mathrm{d}\mathbf{v}}{\mathrm{dt}} = \frac{\mathrm{ke}^{-\mathrm{kt}}}{\mathrm{k}}\mathbf{g} - \mathrm{ke}^{-\mathrm{kt}}\mathbf{u} = \mathrm{e}^{-\mathrm{kt}}\mathbf{g} - \mathrm{ke}^{-\mathrm{kt}}\mathbf{u}$ **M1 A1**

$\mathrm{m}\mathbf{a} = \mathrm{me}^{-\mathrm{kt}}\mathbf{g} - \mathrm{mke}^{-\mathrm{kt}}\mathbf{u} = \mathrm{m}\mathbf{g} - \mathrm{m}\left(1 - \mathrm{e}^{-\mathrm{kt}}\right)\mathbf{g} - \mathrm{mke}^{-\mathrm{kt}}\mathbf{u}$ **M1**

$= \mathrm{m}\mathbf{g} - \mathrm{mk}\left(\frac{1 - \mathrm{e}^{-\mathrm{kt}}}{\mathrm{k}}\mathbf{g} + \mathrm{e}^{-\mathrm{kt}}\mathbf{u}\right) = \mathrm{m}\mathbf{g} - \mathrm{mk}\mathbf{v}$ **M1 re-arrange and substitute A1**

which verifies the equation of motion is satisfied, and

$\mathrm{t} = 0 \Rightarrow \mathbf{r} = \frac{0 - 1 + \mathrm{e}^0}{\mathrm{k}^2}\mathbf{g} + \frac{1 - \mathrm{e}^0}{\mathrm{k}}\mathbf{u} = \mathbf{0}$ and $\mathbf{v} = \frac{1 - \mathrm{e}^0}{\mathrm{k}}\mathbf{g} + \mathrm{e}^0\mathbf{u} = \mathbf{u}$ the initial conditions are satisfied.

B1 (6)

$$\mathrm{r.j} = 0 \Rightarrow -\frac{\mathrm{kT} - 1 + \mathrm{e}^{-\mathrm{kT}}}{\mathrm{k}^2}\,\mathrm{g} + \frac{1 - \mathrm{e}^{-\mathrm{kT}}}{\mathrm{k}}\,\mathrm{u}\sin\alpha = 0$$

M1 substitute

Thus

$$\left(1 - \mathrm{e}^{-\mathrm{kT}}\right)\mathrm{uk}\sin\alpha = \left(\mathrm{kT} - 1 + \mathrm{e}^{-\mathrm{kT}}\right)\mathrm{g}$$

So

$$\mathrm{uk}\sin\alpha = \frac{\left(\mathrm{kT} - 1 + \mathrm{e}^{-\mathrm{kT}}\right)}{\left(1 - \mathrm{e}^{-\mathrm{kT}}\right)}\mathrm{g} = \left(\frac{\mathrm{kT}}{1 - \mathrm{e}^{-\mathrm{kT}}} - 1\right)\mathrm{g}$$

M1 re-arrange *A1 (3)

At time T, it is at the level of projection, and so is descending. **E1**

Thus

$$\tan\beta = \frac{-\mathbf{v}.\mathbf{j}}{\mathbf{v}.\mathbf{i}} = \frac{\frac{1 - \mathrm{e}^{-\mathrm{kT}}}{\mathrm{k}}\,\mathrm{g} - \mathrm{e}^{-\mathrm{kT}}\mathrm{u}\sin\alpha}{\mathrm{e}^{-\mathrm{kT}}\mathrm{u}\cos\alpha} = \frac{\left(\mathrm{e}^{\mathrm{kT}} - 1\right)\mathrm{g}}{\mathrm{uk}\cos\alpha} - \tan\alpha$$

M1 **A1 (3)**

as required.

$$\tan\beta - \tan\alpha = \frac{\left(\mathrm{e}^{\mathrm{kT}} - 1\right)\mathrm{g}}{\mathrm{uk}\cos\alpha} - 2\tan\alpha = \frac{\mathrm{g}}{\mathrm{uk}\cos\alpha}\left(\left(\mathrm{e}^{\mathrm{kT}} - 1\right) - 2\frac{\mathrm{uk}\sin\alpha}{\mathrm{g}}\right)$$

M1

So

$$\tan\beta - \tan\alpha = \frac{\mathrm{g}}{\mathrm{uk}\cos\alpha}\left(\left(\mathrm{e}^{\mathrm{kT}} - 1\right) - 2\left(\frac{\mathrm{kT}}{1 - \mathrm{e}^{-\mathrm{kT}}} - 1\right)\right)$$

M1 substitute

$$= \frac{\mathrm{g}}{\mathrm{uk}\cos\alpha(1 - \mathrm{e}^{-\mathrm{kT}})}\left(\left(\mathrm{e}^{\mathrm{kT}} - 1\right)\left(1 - \mathrm{e}^{-\mathrm{kT}}\right) - 2\left(\mathrm{kT} - \left(1 - \mathrm{e}^{-\mathrm{kT}}\right)\right)\right)$$

$$= \frac{\mathrm{g}}{\mathrm{uk}\cos\alpha(1 - \mathrm{e}^{-\mathrm{kT}})}\left[\mathrm{e}^{\mathrm{kT}} - 1 - 1 + \mathrm{e}^{-\mathrm{kT}} - 2\mathrm{kT} + 2 - 2\mathrm{e}^{-\mathrm{kT}}\right]$$

M1 algebraic manipulation

$$= \frac{2\mathrm{g}}{\mathrm{uk}\cos\alpha(1 - \mathrm{e}^{-\mathrm{kT}})}\left[\frac{\mathrm{e}^{\mathrm{kT}} - \mathrm{e}^{-\mathrm{kT}}}{2} - \mathrm{kT}\right] = \frac{2\mathrm{g}}{\mathrm{uk}\cos\alpha(1 - \mathrm{e}^{-\mathrm{kT}})}(\sinh\mathrm{kT} - \mathrm{kT}) > 0$$

A1 (4)

by the assumption and hence $\tan\beta > \tan\alpha$

Thus as α and β are both acute, $\beta > \alpha$. **M1 A1 (2)**

10. Tension in PX is $\frac{\lambda x}{a}$, tension in QY is $\frac{\lambda y}{a}$, **B1**

and the compression in XY is $\frac{\lambda(x+y)}{a}$. **B1**

So

$$m\frac{d^2x}{dt^2} = -\frac{\lambda x}{a} - \frac{\lambda(x+y)}{a} = -\frac{\lambda(2x+y)}{a}$$

M1 (*) A1 (4)

as required.

$$m\frac{d^2y}{dt^2} = -\frac{\lambda(x+2y)}{a}$$

B1 (1)

So

$$m\frac{d^2(x-y)}{dt^2} = m\left(\frac{d^2x}{dt^2} - \frac{d^2y}{dt^2}\right) = -\frac{\lambda(x-y)}{a}$$

M1

$$\frac{d^2(x-y)}{dt^2} = -\frac{\lambda(x-y)}{ma}$$

i.e.

$$\frac{d^2(x-y)}{dt^2} = -\omega^2(x-y)$$

Thus

$$x - y = A\cos\omega t + B\sin\omega t$$

M1 (*) A1 (3)

$$t = 0\,, x = 0\,, y = -\frac{1}{2}a\,, \frac{dx}{dt} = 0\,, \frac{dy}{dt} = 0$$

M1

So $\frac{1}{2}a = A$, and $0 = \omega B \Longrightarrow B = 0$ **A1 (2)**

That is

$$x - y = \frac{1}{2}a\cos\omega t$$

Similarly

$$\frac{d^2(x+y)}{dt^2} = -\frac{3\lambda(x+y)}{ma} = -3\omega^2(x+y)$$

M1

$$x + y = C\cos\sqrt{3}\omega t + D\sqrt{3}\sin\omega t$$

A1 (2)

$$t = 0\,, x = 0\,, y = -\frac{1}{2}a\,, \frac{dx}{dt} = 0\,, \frac{dy}{dt} = 0$$

So $-\frac{1}{2}a = C$, and $0 = \sqrt{3}\omega D \Longrightarrow D = 0$ **M1**

That is

$$x + y = -\frac{1}{2}a\cos\sqrt{3}\omega t$$

A1 (2)

So

$$y = -\frac{1}{4}a\left(\cos\sqrt{3}\omega t + \cos\omega t\right)$$

M1 A1

To return to the initial position, $y = -\frac{1}{2}a$, and so

$$2 = \cos\sqrt{3}\omega t + \cos\omega t$$

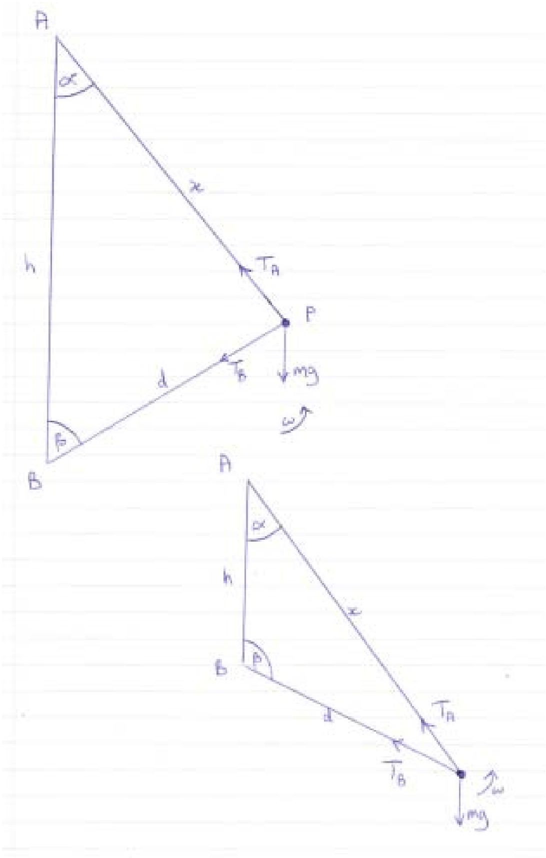
for some t. **M1**

This is only possible if $1 = \cos\sqrt{3}\omega t$ and $1 = \cos\omega t$. **M1**

These require $\sqrt{3}\omega t = 2n\pi$ and $\omega t = 2m\pi$ for non-zero integers n and m. **A1**

For this to occur $\sqrt{3} = \frac{n}{m}$, which is impossible as $\sqrt{3}$ is irrational. Hence it cannot return to its initial position. **E1 (6)**

11.



Resolving vertically

$$T_A \cos \alpha - T_B \cos \beta = mg$$

M1 A1

(or alternatively,

$$T_A \cos \alpha + T_B \cos(180^o - \beta) = mg$$

if other diagram is drawn, which is algebraically equivalent)

Resolving radially inwards

$$T_A \sin \alpha + T_B \sin \beta = m\omega^2 x \sin \alpha$$

M1 A1

(or

$$T_A \sin \alpha + T_B \sin(180^o - \beta) = m\omega^2 x \sin \alpha$$

again algebraically eqivalent)

Solving simultaneously,

$$T_A \cos\alpha \sin\beta + T_A \sin\alpha \cos\beta = mg \sin\beta + m\omega^2 x \sin\alpha \cos\beta$$

M1

$$T_A = m\frac{(g\sin\beta + \omega^2 x \sin\alpha \cos\beta)}{\sin(\alpha+\beta)}$$

A1

and similarly,

$$T_B = m\frac{(\omega^2 x \sin\alpha \cos\alpha - g\sin\alpha)}{\sin(\alpha+\beta)}$$

A1 (7)

As $T_B \geq 0$, $\omega^2 x \sin\alpha\cos\alpha - g\sin\alpha \geq 0$, $(\omega^2 x\cos\alpha - g)\sin\alpha \geq 0$ and as $\sin\alpha > 0$,

M1

$\omega^2 x \cos\alpha - g \geq 0$ i.e. $\omega^2 x\cos\alpha \geq g$ as required. **(*) A1 cso (2)**

By the sine rule, $\frac{d}{\sin\alpha} = \frac{h}{\sin\gamma} \geq h$ as $\sin\gamma \leq 1$, and so $d \geq h\sin\alpha$. **M1 A1**

(Alternatively, the shortest distance of B from the line through AP is that perpendicular to AP which is $h\sin\alpha$, and so $d \geq h\sin\alpha$)

Therefore, $h^2\cos^2\propto = h^2 - h^2\sin^2\alpha \geq h^2 - d^2$ and so $h\cos\propto \geq \sqrt{h^2-d^2}$. **M1 (*) A1 (4)**

If $\omega^2 x\cos\alpha = g$, then

$$T_A = m\frac{(g\sin\beta + \omega^2 x\sin\alpha\cos\beta)}{\sin(\alpha+\beta)} = m\frac{(g\sin\beta + g\tan\propto\cos\beta)}{\sin(\alpha+\beta)} = \frac{mg}{\cos\propto}$$

M1 A1

As $\propto > 0$, $\cos\propto < 1$, and we have shown that $\cos\propto \geq \frac{\sqrt{h^2-d^2}}{h}$. **M1**

So $1 < \frac{1}{\cos\propto} \leq \frac{h}{\sqrt{h^2-d^2}}$, and hence $mg < \frac{mg}{\cos\propto} \leq \frac{mgh}{\sqrt{h^2-d^2}}$ **M1**

i.e. $mg < T_A \leq \frac{mgh}{\sqrt{h^2-d^2}}$ **A1 (5)**

Equality in the upper bound occurs when $\sin\gamma = 1$, that is when AP and BP are perpendicular.

M1 A1 (2)

12. (i) $P(X < x_m) = \frac{1}{2}$ so $P(e^X < e^{x_m}) = \frac{1}{2}$ i.e. $P(Y < e^{x_m}) = \frac{1}{2}$ **M1**

But $P(Y < y_m) = \frac{1}{2}$ and so $y_m = e^{x_m}$ **A1 (2)**

(ii) $P(Y < y) = P(e^X < y) = P(X < \ln y) = \int_0^{\ln y} f(x)\,dx$ **M1 A1**

Substituting $x = \ln t$, $\int_0^{\ln y} f(x)\,dx = \int_{-\infty}^{y} f(\ln t)\,\frac{1}{t}dt$ so the probability density function of Y is $\frac{f(\ln y)}{y}$ where $-\infty < y < \infty$. **M1 A1 (4)**

For the mode λ of Y, $\frac{d}{dy}\left(\frac{f(\ln y)}{y}\right) = 0$ when $y = \lambda$. **M1**

Thus, $\frac{yf'(\ln y)\frac{1}{y} - f(\ln y)}{y^2} = 0$ when $y = \lambda$ and hence $f'(\ln\lambda) = f(\ln\lambda)$ **M1 (*) A1 (3)**

(iii) $\frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu-\sigma^2)^2/(2\sigma^2)}$ is the pdf of $X \sim N(\mu + \sigma^2, \sigma^2)$ and hence

$$\frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(x-\mu-\sigma^2)^2/(2\sigma^2)}\,dx = 1$$

E1 (1)

$E(Y) = \int_0^{\infty} y\frac{f(\ln y)}{y}dy$ where $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$

So $E(Y) = \frac{1}{\sigma\sqrt{2\pi}}\int_0^{\infty} e^{-(\ln y-\mu)^2/(2\sigma^2)}\,dy = \frac{1}{\sigma\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-(t-\mu)^2/(2\sigma^2)}\,e^t dt$ using the substitution $y = \ln t$ **M1 A1**

The exponent of e in this integral is $\frac{-(t-\mu)^2}{(2\sigma^2)} + t = \frac{-(t^2-2\mu t+\mu^2-2\sigma^2 t)}{(2\sigma^2)} = \frac{-(t^2-2t(\mu+\sigma^2)+\mu^2)}{(2\sigma^2)} = \frac{-\left(t-(\mu+\sigma^2)\right)^2+2\mu\sigma^2+\sigma^4}{(2\sigma^2)}$ **M1**

Thus the required integral, by the explained result equals $e^{\frac{2\mu\sigma^2+\sigma^4}{(2\sigma^2)}} = e^{\mu+\frac{1}{2}\sigma^2}$ **(*) A1 (4)**

(iv) Using the result from (ii), with $f(x) = \frac{1}{\sigma\sqrt{2\pi}} e^{-(x-\mu)^2/(2\sigma^2)}$,

$$\frac{1}{\sigma\sqrt{2\pi}}\cdot -\frac{(\ln\lambda-\mu)}{\sigma^2} e^{-(\ln\lambda-\mu)^2/(2\sigma^2)} = \frac{1}{\sigma\sqrt{2\pi}} e^{-(\ln\lambda-\mu)^2/(2\sigma^2)}$$

M1 A1

That is $-(\ln \lambda - \mu) = \sigma^2$, and so $\lambda = e^{\mu - \sigma^2}$ **A1 (3)**

As $x_m = \mu$ **M1** we have from (i) $y_m = e^{x_m} = e^{\mu}$ **A1** and from (iii) $E(Y) = e^{\mu + \frac{1}{2}\sigma^2}$ **(2)**

$e^{\mu - \sigma^2} < e^{\mu} < e^{\mu + \frac{1}{2}\sigma^2}$ and so $\lambda < y_m < E(Y)$ **E1 (1)**

13. (i) $P(N=0)=1$ and $P(N=r\ where\ r\neq 0)=0$, **M1**

So $PGF=P(N=0)+tP(N=1)+t^2P(N=2)+\cdots=1$ **A1 (2)**

(ii) $PGF=P(N=0)+tP(N=1)+t^2P(N=2)+\cdots$ and this is exactly the same as at the start of the whole game as nothing was scored in the first round, i.e. $G(t)$ **M1 A1 (2)**

(iii) The pgf conditional on increasing the score by one and continuing to the next round is $tG(t)$ as the probability of each total score plus one is whatever the probability of the total score was before the first round. **M1 A1 (2)**

Hence, as one if these three things must happen on the first round, **E1 (1)**

$G(t)=a\times 1+b\times G(t)+c\times tG(t)$

Thus $G(t)-bG(t)-ctG(t)=a$, and so, $G(t)=a(1-b-ct)^{-1}$ **M1 A1**

$$G(t)=a(1-b-ct)^{-1}=a(1-b)^{-1}\left(1+\left(\frac{-ct}{1-b}\right)\right)^{-1}$$

The coefficient of t^n is $a(1-b)^{-1}.\frac{-1.-2.\ldots.-n}{n!}.\left(\frac{-c}{1-b}\right)^n=\frac{ac^n}{(1-b)^{n+1}}$ so $P(N=n)=\frac{ac^n}{(1-b)^{n+1}}$ as

M1 A1 **(*) A1 (5)**

required.

(iv) $\mu=E(N)=G'(1)$ **M1**

$G(t)=a(1-b-ct)^{-1}$ so $G'(t)=ac(1-b-ct)^{-2}$, and so **M1 A1**

$\mu=G'(1)=ac(1-b-c)^{-2}=aca^{-2}=c/a$ **A1** **(4)**

So $c=\mu a$ and thus $b=1-a-c=1-a-\mu a$, so **M1 A1**

$P(N=n)=\frac{ac^n}{(1-b)^{n+1}}=\frac{a(\mu a)^n}{(1-(1-a-\mu a))^{n+1}}=\frac{\mu^n a^{n+1}}{(a+\mu a)^{n+1}}=\frac{\mu^n a^{n+1}}{a^{n+1}(1+\mu)^{n+1}}=\frac{\mu^n}{(1+\mu)^{n+1}}$ as required.

M1 **(*) A1 (4)**