S2 2014 Report

General Comments

There were good solutions presented to all of the questions, although there was generally less success in those questions that required explanations of results or the use of diagrams and graphs to reach the solution. Algebraic manipulation was generally well done by many of the candidates although a range of common errors such as confusing differentiation and integration and simple arithmetic slips were evident. Candidates should also be advised to use the methods that are asked for in questions unless it is clear that other methods will be accepted (such as by the use of the phrase “or otherwise”).

Comments on individual questions.

Question 1

While the first part of the question was successfully completed by many of the candidates, there were quite a few diagrams drawn showing the point P further from the line AB than Q. Those who established the expression for $x\cos\theta$ were usually able to find an expression for $x\sin\theta$ and good justifications of the quadratic equation were given. The case where P and Q lie on the lines AC produced and BC produced caused a lot of difficulty for many of the candidates, many of whom tried unsuccessfully to create an argument based on similar triangles.

The condition for (*) to be linear in x did not cause much difficulty, although a number of candidates did not give the value of $\cos^{-1}\left(-\frac{1}{2}\right)$. Many candidates realised that the justification that the roots were distinct would involve the discriminant, although some solutions included the case where the discriminant could be equal to 0 were produced. However, very few solutions were able to give a clear justification that the discriminant must be greater than 0.

In the final part some candidates sketched the graph of the quadratic rather than sketching the triangle in the two cases given. In the second case many candidates did not realise that Q was at the same point as C.

Question 2

This was one of the more popular questions of the paper. Most candidates successfully showed that the first inequality was satisfied, but when producing counterexamples, some failed to show that either $f(x) \neq 0$ or $f(\pi) \neq 0$ for their chosen functions. In the second part many candidates did not attempt to choose values of a, b and c, but substituted the general form of the quadratic function into the inequality instead. In the case where the function involved trigonometric functions, many of those who attempted it were able to deduce that $p = q = -r$, but several candidates made mistakes in the required integration. Those who established two inequalities were able to decide which gives the better estimate for π.

Question 3

Many candidates produced a correct solution to the first part of the question. There were a number of popular methods, such as the use of similar triangles, but an algebraic approach finding the intersection between the line and a perpendicular line through the origin was the most popular. Some candidates however, simply stated a formula for the shortest distance from a point to a line. Establishing the differential equation in the second part of the question was generally done well, but many candidates struggled with the solution of the differential equation. A common error was to ignore the case $y'' = 0$ and simply find the circle solution.

The final part of the question was attempted by only a few of the candidates, many of whom did not produce an example that satisfied all of the conditions stated in the question, in particular the condition that the tangents should not be vertical at any point was often missed.

Question 4

Many candidates were able to perform the given substitution correctly and then correctly explain how this demonstrates that the integral is equal to 1. The second part caused more difficulty, particularly with candidates not able to state the relationship between $\arctan x$ and $\arctan\left(\frac{1}{x}\right)$. Attempts to integrate with the substitution $v = \arctan\left(\frac{1}{u}\right)$ often resulted in an incorrect application of the chain rule when finding $\frac{dv}{du}$.

In the final part of the question many candidates attempted to use integration by parts to reach the given answer.

Question 5

This was the most popular question on the paper and the question which had the highest average score. Most candidates correctly solved the differential equation in the first part of the question, but many then calculated the constant term incorrectly. In the second part of the question most candidates were able to find the appropriate values of a and b, but then did not see how to apply the result from part (i) and so did the integration again or just copied the answer from the first part. Some candidates again struggled to obtain the correct constant for the integration and others did not substitute the correct values for the point on the curve (taking (X, Y) as $(1,1)$ rather than (x, y)).

Question 6

This was one of the less popular of the pure maths questions, but the average mark achieved on this paper was one of the highest for the paper. The first section did not present too much difficulty for the majority of candidates, with a variety of methods being used to show the first result such as proof by induction or use of $e^x = \cos x + i \sin x$. In the second part of the question many of the candidates struggled to explain the reasoning clearly to show the required result. Most candidates who reached the final part of the question realised that the previous part provides the basis for a proof by induction.

Question 7

This was another of the less popular pure maths questions. The nature of this question meant that many solutions involved a series of sketches of graphs with very little written explanation. Most candidates were able to identify that the sloping edges of $y = f(x)$ would have the same gradient as the sloping edges of $y = g(x)$, but many did not have both sloping edges overlapping for the two graphs. In some cases only one sloping edge of $y = g(x)$ was drawn. A large number of candidates who correctly sketched the graphs identified the quadrilateral as a rectangle, rather than a square. In the second part of the question, sketches of the case with one solution often did not have the graph of $y = |x - c|$ meeting the x-axis at one corner of the square identified in part (i), although many candidates were able to identify the different cases that could occur. Unfortunately in the final part of the question very few candidates used the result from the first part of the question and so considered a number of possibilities that do not exist for any values of a, b, c and d.

Question 8

This was the least popular of the pure maths questions and also the one with the lowest average score. Many of the candidates were able to show the required result at the start of the question, although very few candidates explained that m could be either of the two integers when the range included two integers. Parts (i) and (ii) were then quite straightforward for most candidates, although many calculated the range of values but did not justify their choice in the case where there were two possibilities. In the final two parts of the question some candidates mistakenly chose the value 0 when asked for a positive integer.

Question 9

This question was not attempted by a very large number of candidates and the average score achieved was the lowest on the paper. While there were a number of attempts that did not proceed beyond drawing a diagram to represent the situation, the first part of the question was done well by a large number of candidates. Many were also able to adjust the result for the case when the frictional force acts downwards. Unfortunately, in the final part of the question many candidates continued to use $F = \mu R$, not realising that this only applies in the critical case and so there were very few correct solutions to this part of the question.

Question 10

This was the most popular of the mechanics questions and also the one that had the best average score, although candidates did struggle to get very high marks on the question particularly on the final parts. The first part of the question asks for a derivation of the equation for the trajectory which was familiar to many candidates, although in some cases the result was obtained by stating that it is a parabola and knowledge of the maximum value and the range. Many candidates who successfully obtained the Cartesian equation then struggled with the differentiation with respect to λ, instead finding the maximum height for a constant value of λ. Unfortunately, this made the remainder of the question insoluble. Some candidates decided to differentiate with respect to θ instead, which did not cause any serious problems, although it did require more work. A few candidates used the discriminant rather than differentiation, but did not provide any justification of this method.

Candidates were able to draw the graph, but many did not label the area that was asked for in the question. Those who reached the final part of the question and considered the distance function for the position during the flight used differentiation to work out the greatest distance. However, many did not realise that the maximum value of a function can be achieved at an end-point of the domain even with a derivative that is non-zero.

Question 11

Many candidates who attempted this question struggled, particularly due to a difficulty in drawing a diagram to represent the situation. From these incorrect diagrams candidates often reached results where one of the signs did not match that given in the question. The calculation of the acceleration was found to be difficult by many of the candidates, although those who understood that differentiation of the coordinates of P would give the acceleration were then able to complete the rest of the question correctly. Those candidates that attempted the final part of the question were able to solve it correctly.

Question 12

This was the least popular question on the paper. A large number of candidates who attempted this question seemed unable to work out where to start on the first part of the question. Much of the rest of the question requires working with the hazard function defined at the start of the question and so many candidates who attempted these parts were able to do the necessary integration to solve the differential equations that arose. A common error among those who attempted part (iv) was to ignore the “if and only if” statement in the question and only show the result one way round.

Question 13

This was the more popular of the two questions on Probability and Statistics, but as in previous years it still only attracted answers from a very small number of candidates. The average mark for this question was also quite low, often due to a difficulty in explaining the reasoning behind some of the parts of the question. Many candidates were able to find the expression for $P(X = 4)$ and most were then able to obtain the general formula required in part (i) of the question, although a number of candidates did not include the correct number of factors in the answer. Parts (ii) and (iii) did not cause too much difficulty, but the final part required a clear explanation to gain full marks.

STEP II 2014

Question 1:

Drawing a diagram and considering the horizontal and vertical distances will establish the relationships for $x\cos\theta$ and $x\sin\theta$ easily. The quadratic equation will then follow from use of the identity $\cos^2\theta + \sin^2\theta \equiv 1$. The same reasoning applied to a diagram showing the case where P and Q lie on AC produced and BC produced will show that the same equation is satisfied.

(*) will be linear if the coefficient of x^2 is 0, so therefore $\cos(\alpha+\beta)$ will need to equal $-\frac{1}{2}$, which gives a relationship between α and β. For (*) to have distinct roots the discriminant must be positive. Using some trigonometric identities it can be shown that the discriminant is equal to $4(1-(\sin\alpha - \sin\beta)^2)$ and it should be easy to explain why this must be greater than 0.

The first case in part (iii) leads to $x = \sqrt{2} \pm 1$ and so there are two diagrams to be drawn. In each case the line joining P to Q will be horizontal.

The second case in part (iii) is an example where (*) is linear. This leads to $x = \frac{\sqrt{3}}{3}$. Therefore Q is at the same point as C and so the point P is the midpoint of AC.

Question 2:

By rewriting in terms of $\cos 2nx$ it can be shown that $\int_0^\pi \sin^2 nx\, dx = \frac{\pi}{2}$ and $\int_0^\pi n^2\cos^2 nx\, dx = \frac{n^2\pi}{2}$. Therefore (*) must be satisfied as n is a positive integer. The function $f(x) = x$ does not satisfy (*) and $f(0) = 0$ but $f(\pi) \neq 0$. The function $g(x) = f(\pi - x)$ will therefore provide a counterexample where $g(\pi) = 0$, but $g(0) \neq 0$.

In part (ii), $f(x) = x^2 - \pi x$ will need to be selected to be able to use the assumption that (*) is satisfied. The two sides of (*) can then be evaluated:

$$\int_0^\pi x^4 - 2\pi x^3 + \pi^2 x^2\, dx = \frac{\pi^5}{30}$$

$$\int_0^\pi 4x^2 - 4\pi x + \pi^2\, dx = \frac{\pi^3}{3}$$

Substitution into (*) then leads to the inequality $\pi^2 \leq 10$.

To satisfy the conditions on $f(x)$ for the second type of function, the values of p, q and r must satisfy $q + r = 0$ and $p + r = 0$. Evaluating the integrals then leads to $\pi \leq \frac{22}{7}$.

Since $\left(\frac{22}{7}\right)^2 < 10$, $\pi \leq \frac{22}{7}$ leads to a better estimate for π^2.

Question 3:

By drawing a diagram and marking the shortest distance a pair of similar triangles can be used to show that $\frac{c/m}{c\sqrt{m^2+1}/m} = \frac{d}{c}$, which simplifies to $d = c/\sqrt{m^2+1}$.

For the second part, the tangent to the curve at the general point (x, y) will have a gradient of y' and so the y-intercept will be at the point $(0, y - xy')$. Therefore the result from part (i) can be applied using $m = y'$ and $c = y - xy'$ to give $a = (y - xy')/\sqrt{(y')^2 + 1}$, which rearranges to give the required result.

Differentiating the equation then gives $y''\left(a^2y' + x(y - xy')\right) = 0$ and so either $y'' = 0$ or $a^2y' + x(y - xy') = 0$.

If $y'' = 0$ then the equation will be of a straight line and the y-intercept can be deduced in terms of m.

If $a^2y' + x(y - xy') = 0$, then the differential equation can be solved to give the equation of a circle.

Part (iii) then requires combining the two possible cases from part (ii) to construct a curve which satisfies the conditions given. This must be an arc of a circle with no vertical tangents, with straight lines at either end of the arc in the direction of the tangents to the circle at that point.

Question 4:

In part (i), if the required integral is called I then the given substitution leads to an integral which can be shown to be equal to $-I$. This means that $2I = 0$ and so $I = 0$.

In part (ii), once the substitution has been completed, the integral will simplify to $\int_{1/b}^{b} \frac{\arctan\frac{1}{u}}{u} du$. Since $\arctan x + \arctan\left(\frac{1}{x}\right) = \frac{\pi}{2}$ the integral can be shown to be equal to $\frac{1}{2}\int_{1/b}^{b} \frac{\pi}{2x} dx$, which then simplifies to the required result.

In part (iii), making with the substitution in terms of k and simplifying will show that the integral is equivalent to

$$\int_0^\infty \frac{ku^2}{(a^2u^2 + k^2)^2} du$$

Therefore choosing $k = a^2$, the integral can be simplified further to

$$\frac{1}{a^2}\int_0^\infty \frac{u^2}{(a^2 + u^2)^2} du = \frac{1}{a^2}\int_0^\infty \frac{1}{a^2 + u^2} du - \frac{1}{a^2}\int_0^\infty \frac{a^2}{(a^2 + u^2)^2} du$$

The result then follows by using the given value for $\int_0^\infty \frac{1}{a^2+x^2} dx$.

Question 5:

Using the substitution $y = xu$, the differential equation can be simplified to

$$x\frac{du}{dx} = \frac{1 + 4u - u^2}{u - 2}$$

This can be solved by separating the variables after which making the substitution $u = \frac{y}{x}$ and substituting the point on the curve gives the required quadratic in x and y.

In part (ii), $\frac{dY}{dX}$ can be shown to be equal to $\frac{dy}{dx}$. The values of a and b need to be chosen so that the right hand side of the differential equation has no constant terms in the numerator or denominator. This leads to the simultaneous equations:

$$a - 2b - 4 = 0$$

$$2a + b - 3 = 0$$

Solving these and substituting the values into the differential equation gives $\frac{dY}{dX} = \frac{X-2Y}{2X+Y}$, and so

$$\frac{dX}{dY} = \frac{2X + Y}{X - 2Y}$$

This is the same differential equation as in part (i), with $x = Y$ and $y = X$. Most of the solution in part (i) can therefore be applied, but the point on the curve is different, so the constant in the final solution will need to be calculated for this case.

Question 6:

One of the standard trigonometric formulas can be used to show that

$$\sin\left(r + \tfrac{1}{2}\right)x - \sin\left(r - \tfrac{1}{2}\right)x = 2\cos rx \sin\tfrac{1}{2}x.$$

Summing these from $r = 1$ to $r = n$ will then give the required result.

In part (i), the definition can be rewritten as $S_2(x) = \sin x + \frac{1}{2}\sin 2x$. The stationary points can then be evaluated by differentiating the function. The sketch is then easy to complete.

For part (ii), differentiating the function gives $S_n'(x) = \cos x + \cos 2x + \cdots + \cos nx$. Applying the result from the start of the question, this can be written as

$$S_n'(x) = \frac{\sin\left(n+\frac{1}{2}x\right) - \sin\frac{1}{2}x}{2\sin\frac{1}{2}x}$$

Since $\sin\frac{1}{2}x \neq 0$ in the given range, the stationary points are where $\sin\left(n+\frac{1}{2}\right)x - \sin\frac{1}{2}x = 0$. This can then be simplified to the required form by splitting $\sin\left(n+\frac{1}{2}\right)x$ into functions of nx and $\frac{1}{2}x$ and noting that $\sin\frac{1}{2}x \neq 0$ and $\cos\frac{1}{2}x \neq 0$ in the given range, so both can be divided by. By noting that the difference between $S_{n-1}(x)$ and $S_n(x)$ is $\frac{1}{n}\sin nx$ the result just shown can be used to show the final result of part (ii). Part (iii) then follows by induction.

Question 7:

By considering the regions $x \leq a$, $a < x < b$ and $x \geq b$, $f(x)$ can be written as

$$f(x) \quad = \quad \begin{matrix} a + b - 2x & x \leq a \\ b - a & a < x < b \\ 2x - a - b & x \geq b \end{matrix}$$

Therefore the graph of $y = f(x)$ will be made up of two sloping sections (with gradients 2 and -2 and a horizontal section). The graph of $y = g(x)$ will have the same definition in the regions $x \leq a$ and $x \geq b$, with the sloping edges extending to a point of intersection on the x-axis. The quadrilateral with therefore have sides of equal length and right angles at each vertex, so it is a square.

In part (ii), sketches of the cases where $c = a$ and $c = b$ show that these cases give just one solution. If $a < c < b$ there will be no solutions and in the other regions there will be two solutions.

In part (iii) the graphs for the two sides of the equation can be related to graphs of the form of $g(x)$ (apart from the section which is replaced by a horizontal line) in the first part of the question. Since $d - c < b - a$, the horizontal sections of the two graphs must be at different heights so the number of solutions can be seen to be the same as the number of intersections of the graphs of the form of $g(x)$.

Question 8:

The coefficients from the binomial expansion should be easily written down. It can then be shown that

$$\frac{c_{r+1}}{c_r} = \frac{b(n-r)}{a(r+1)}$$

This will be greater than 1 (indicating that the value of c_r is increasing) while $b(n-r) > a(r+1)$, which simplifies to $r < \frac{nb-a}{a+b}$. Similarly, $\frac{c_{r+1}}{c_r} = 1$ if $r = \frac{nb-a}{a+b}$ and $\frac{c_{r+1}}{c_r} < 1$ if $r > \frac{nb-a}{a+b}$. Therefore the maximum value of c_r will be the first integer after $\frac{nb-a}{a+b}$ (and there will be two maximum values for c_r if $\frac{nb-a}{a+b}$ is an integer. The required inequality summarises this information.

In parts (i) and (ii) the values need to be substituted into the inequality. Where there are two possible values, it needs to be checked that they are equal before taking the higher if this has not been justified in the first case.

In part (iii) the greatest value will be achieved when the denominator takes the smallest possible value, therefore $a = 1$, and then in part (iv) the greatest value will be achieved by maximising the numerator. Since the maximum possible value of $G(n, a, b)$ is n, $b \geq n$ will achieve this maximum.

Question 9:

Once a diagram has been drawn the usual steps will lead to the required result:

Resolving vertically:

$$F + T\cos\theta = mg$$

Resolving horizontally:

$$T\sin\theta = R$$

Taking moments about A:

$$mg(a\cos\varphi + b\sin\varphi) = Td\sin(\theta + \varphi)$$

Limiting equilibrium, so $F = \mu R$:

$$\mu T\sin\theta + T\cos\theta = mg$$

Therefore:

$$Td\sin(\theta + \varphi) = T(\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

And so:

$$d\sin(\theta + \varphi) = (\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

If the frictional force were acting in the opposite direction, then the only change to the original equations would be the sign of F in the first equation. Therefore the final relationship will change to

$$d\sin(\theta + \varphi) = (-\mu\sin\theta + \cos\theta)(a\cos\varphi + b\sin\varphi)$$

For the final part, the first and third of the equations above can be used to show that

$$F = \frac{Td\sin(\theta + \varphi)}{a\cos\varphi + b\sin\varphi} - T\cos\theta$$

Since $F > 0$ if the frictional force is upwards, this then leads to the condition $d > \frac{a+b\tan\varphi}{\tan\theta+\tan\varphi}$. Since the string must be attached to the side AB, d cannot be bigger than $2b$, which leads to the final result of the question.

Question 10:

Consideration of the motion horizontally and vertically and eliminating the time variable leads to a Cartesian equation for the trajectory:

$$y = \lambda x - \frac{gx^2}{2u^2}(1 + \lambda^2)$$

The maximum value can be found either by differentiation or by completing the square. Completing the square gives:

$$y = -\frac{gx^2}{2u^2}\left(\lambda - \frac{u^2}{gx}\right) + \frac{u^2}{2g} - \frac{gx^2}{2u^2}$$

Which shows that $Y = \frac{u^2}{2g} - \frac{gx^2}{2u^2}$. If this graph is sketched then the region bounded by the graph and the axes will represent all the points that can be reached.

The maximum achievable distance must lie on the curve and the distance, d, of a point on the curve can be shown to satisty $d^2 = \left(\frac{u^2}{2g} + \frac{gx^2}{2u^2}\right)^2$, which must be maximised when x takes the maximum value possible.

Question 11:

A diagram shows that the coordinates of P are $(x + (L - x)\sin\alpha, -(L - x)\cos\alpha)$

Therefore, by differentiating the y-coordinate of P shows that the vertical acceleration of P is $\ddot{x}\cos\alpha$ and applying Newton's Second Law gives

$$T\cos\alpha - kmg = km\ddot{x}\cos\alpha$$

A similar method for the horizontal motion of P and R gives the two equations

$$T\sin\alpha = -km(1 - \sin\alpha)\ddot{x}$$

$$T - T\sin\alpha = -m\ddot{x}$$

For part (ii), eliminating T from the last two equations gives the required relationship. A sketch of the graph of $y = \frac{x}{(1-x)^2}$ will then show that for any value of k there is a possible value between 0 and 1 for $\sin\alpha$.

In part (iii), elimination of T from the two equations formed by considering the motion of P gives the required result.

Question 12:

The required probability in the first part is given by

$$\frac{P(t < T < t + \delta t)}{P(T > t)} = \frac{F(t + \delta t) - F(t)}{1 - F(t)}$$

In the case of small values of δt, $F(t + \delta t) - F(t) \approx f(t)\delta t$, which leads to the correct probability.

In part (ii), differentiation gives $f(t) = \frac{1}{a}$, and substituting into the definition of the hazard function gives $h(t) = \frac{1}{a-t}$. Both graphs are simple to sketch.

In part (iii), using the definition of the hazard function gives $\frac{F\prime(t)}{1-F(t)} = \frac{1}{t}$. Integrating gives $-\ln|1 - F(t)| = \ln|kt|$, and so the probability density function can be found by rearranging to find $F(t)$ and then differentiating.

A similar method in part (iv) shows that if $h(t)$ is of the form stated then $f(t)$ will be of the given form. Similarly, if $f(t)$ has the given form then $h(t)$ can be shown to have the form stated.

In part (v), a differential equation can again be written using the definition of the hazard function and this can again be solved by integrating both sides with respect to t.

Question 13:

Considering the sequence of events for $X = 4$, the 1st, 2nd and 3rd numbers must all be different and then the 4th must be the same as one of the first three. The probability is therefore

$$P(X = 4) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n}$$

The same reasoning applied to $X = r$ gives

$$P(X = r) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{r-2}{n}\right)\frac{r-1}{n}$$

The result of part (i) is then found by observing that the probabilities of all possible outcomes add up to 1.

Substituting the probabilities into the formula for $E(X)$ gives

$$E(X) = \frac{2}{n} + 3\left(1 - \frac{1}{n}\right)\frac{2}{n} + 4\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots + (n+1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)$$

For part (iii) observe that any case where $X \geq k$ will have the first $k - 1$ numbers all different from each other. Therefore

$$P(X \geq k) = \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{k-2}{n}\right)$$

The first formula in part (iv) can be shown by considering $kP(Y = k)$ to be equal to the sum of k copies of $P(Y = k)$ and then regrouping the sum for $E(Y)$. Finally this gives two different expressions for $E(Y)$, which must be equal to each other:

$$\begin{aligned}\frac{2}{n} + 3\left(1 - \frac{1}{n}\right)\frac{2}{n} + 4\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\frac{3}{n} + \cdots + (n+1)\left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right) \\ = 1 + 1 + \left(1 - \frac{1}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right) + \left(1 - \frac{1}{n}\right)\left(1 - \frac{2}{n}\right)\cdots\left(1 - \frac{n-1}{n}\right)\end{aligned}$$

Rearranging and using the result from part (i) then gives the required result.