# STEP CORRESPONDENCE PROJECT

## Assignment 12

## Warm-up

**1** In this question, $n$ denotes a positive integer.

**(i)** Show that $n(n+1)$ is divisible by 2.

Show that $n(n+1)(n+2)$ is divisible by 3.

**(ii)** Factorise completely $n^3 - n$, and deduce that it is divisible by 6.

**(iii)** Show that $n^5 - n^3$ is divisible by 24. You may want to consider $n$ odd and $n$ even separately.

**(iv)** Show that $2^{2n} - 1$ is divisible by 3.

**(v)** Show that, if $n-1$ is divisible by 3, then $n^3 - 1$ is divisible by 9. For this part, factorising is not the easiest method; instead, you could try writing $n - 1 = 3k$ (where $k$ is an integer).

## Preparation

**2** **(i)** Express the following as single fractions:

**(a)** $\dfrac{1}{(x-1)(x+2)} - \dfrac{1}{(x+1)(x+2)}$

**(b)** $\dfrac{m}{m+2} \times \dfrac{m-1}{m+1} + \dfrac{m}{m+2} \times \dfrac{2}{m+1} \times \dfrac{m-1}{m}$

**(ii)** I have a bag of sweets, which contains 9 mint imperials and 6 lemon sherbets. I take two sweets out of the bag without looking (one after the other) and eat them. What is the probability that I eat two sweets of the same flavour?

**(iii)** I have another bag of sweets with $a$ apple sours and $b$ blackcurrant chews.

**(a)** If I take one sweet at random, what is the probability that it is an apple sour?

**(b)** If I take three sweets one after the other and eat them what is the probability that they are all blackcurrant chews? (Don't attempt to simplify: leave your answer as the product of three fractions.)

**(iv)** Three children have a swimming lesson. Each child has a probability $\frac{1}{4}$, independently of the other two, of remembering to bring his or her goggles to the lesson. By considering the different possibilities find the probability that:

**(a)** at least one child has goggles;

**(b)** the first child in the queue has goggles;

**(c)** the middle child has not got goggles, but the other two have.

## The STEP question

**3** I am selling raffle tickets for £1 per ticket. In the queue for tickets, there are $m$ people each with a single £1 coin and $n$ people each with a single £2 coin. Each person in the queue wants to buy a single raffle ticket and each arrangement of people in the queue is equally likely to occur. Initially, I have no coins and a large supply of tickets. I stop selling tickets if I cannot give the required change.

**(i)** In the case $n = 1$ and $m \geqslant 1$, find the probability that I am able to sell one ticket to each person in the queue.

**(ii)** By considering the first three people in the queue, show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 2$ and $m \geqslant 2$ is $\dfrac{m-1}{m+1}$.

**(iii)** Show that the probability that I am able to sell one ticket to each person in the queue in the case $n = 3$ and $m \geqslant 3$ is $\dfrac{m-2}{m+1}$.

## Discussion

In parts (ii) and (iii) the answers are given, so you must **carefully** justify your answers. Set your work out so that the different possibilities for the queues appear in some logical order.

For part (iii), you will need to consider more than the first 3 people. You will need to persevere: you end up having to add up four or five long fractions (you can do some cancelling before you start adding).

You might have noticed a pattern in these three results, from which you might conjecture that the probability of being able to sell one ticket to each person in the general case $m \geqslant n$ is $\dfrac{m+1-n}{m+1}$. This turns out to be correct, though the proof is significantly more difficult than these special cases.

## Warm down

**4** Five children (Anna, Bryan, Charlie, Daniel and Emily) raced each other. First they raced to the spreading chestnut tree, and then they raced back to their starting point. The following facts are known:

**(i)** Anna was fourth in the race to the tree.

**(ii)** The person who was last to the tree managed to win the race back.

**(iii)** The person who won the race to the tree was third on the way back.

**(iv)** The person who was third in the race to the tree was second on the way back.

**(v)** Bryan was fourth on the way back.

**(vi)** Charlie reached the tree before Daniel.

**(vii)** Charlie got back to the start before Emily.

For each race (to the tree and back again), write down the order in which the children finished.

You might want to use a table to display the results, giving names such as Yolanda (or letters such as $Y$) to people you do not yet know but about whom you have information. (So from part (ii), you could say that $Y$ came 5th in the first race and 1st in the second race.)