STEP CORRESPONDENCE PROJECT

Assignment 13

Warm-up

1 We define two new functions C and S by

$$\mathrm{C}(x) = \tfrac{1}{2}\big(a^x + a^{-x}\big) \quad \text{and} \quad \mathrm{S}(x) = \tfrac{1}{2}\big(a^x - a^{-x}\big)$$

where a is some fixed positive real number.

(i) Show, using these definitions, that

(a) $\big(\mathrm{C}(x)\big)^2 - \big(\mathrm{S}(x)\big)^2 = 1$
(b) $\mathrm{C}(x)\mathrm{C}(y) + \mathrm{S}(x)\mathrm{S}(y) = \mathrm{C}(x+y)$
(c) $\mathrm{C}(x)\mathrm{S}(y) + \mathrm{S}(x)\mathrm{C}(y) = \mathrm{S}(x+y)$.

Deduce an expression for $\mathrm{C}(2x)$ in terms of $\mathrm{C}(x)$.

(ii) *You are given* that the limit as $h \to 0$ of $\dfrac{a^h - 1}{h}$ is K, i.e.

$$\lim_{h\to 0} \frac{a^h - 1}{h} = K\,,$$

where K some given number[1].

Show from first principles, using the definition of differentiation, that

$$\frac{\mathrm{d}(a^x)}{\mathrm{d}x} = Ka^x$$

and that

$$\frac{\mathrm{d}(a^{-x})}{\mathrm{d}x} = -Ka^{-x}$$

Note: $\dfrac{\mathrm{d}(a^x)}{\mathrm{d}x} \neq xa^{x-1}$.

(iii) Hence (even if you didn't manage to do part (ii) above) find the derivatives of $\mathrm{C}(x)$ and $\mathrm{S}(x)$ and deduce that $\mathrm{C}(x)$ satisfies the differential equation

$$\frac{\mathrm{d}^2\mathrm{C}(x)}{\mathrm{d}x^2} = K^2\,\mathrm{C}(x)\,.$$

You are not asked to *solve* this equation: you just have to substitute $\mathrm{C}(x)$ into the left hand side of the equation and check that it gives the right hand side.

[1] Actually, $K = \ln a$ i.e. $\log_e a$, but we haven't used this because some of you might not yet have met natural logs or the number e.

Preparation

You may have already met the modulus function, $|x|$. In case you have not, it is defined as follows:

$$|x| = \begin{cases} +x & \text{for } x \geqslant 0 \\ -x & \text{for } x < 0 \end{cases}$$

The effect of the modulus function is to make the argument of the function (i.e. the input) positive. Removing modulus signs gives either the positive or negative value of the argument. For example, if $|x| = 3$ then either $x = 3$ or $x = -3$.

2 **(i)** Solve the equation $|2x - 3| - 4 = 3$

Start by writing this in the form $|y| = c$. Then you have two equations to solve: $y = c$ and $y = -c$.

(ii) Sketch the graph $y = |2x - 3|$

There are two usual ways to proceed.

One way is to look at the "critial value" of x, where $|2x - 3| = 0$, and split the graph into two regions. In one region $y = 2x - 3$ and in the other region $y = -(2x - 3)$.

The other way is to sketch the graph of $y = 2x - 3$ and then reflect the part that lies under the x-axis about the x-axis (so all the graph now lies above the x-axis).

(iii) Solve the equation $|2x| + |x - 1| = 3$

There are two "critical values" here, at $x = 0$ and $x = 1$.

If $x < 1$ then $|x - 1| = -(x - 1)$ (in order to make it positive), and if $x > 1$ then $|x - 1| = x - 1$. You need to consider three different equations for the three regions separated by the critical values. You must make sure that any solutions you find actually fall in the region you are considering. Substituting your values back into the original equation to check that they work is always a good idea.

(iv) Sketch the graph $|x| + |y| = 1$ in the regions of the x-y plane given by:

- $x > 0$ and $y > 0$;
- $x > 0$ and $y < 0$.

For the first region, $|x| = x$ and $|y| = y$ so the graph is just $x + y = 1$

(v) Sketch the graph $|x - 1| + |y - 1| = 1$ in the region where $x < 1$ and $y > 1$. In this region, shade the subset of the x-y plane in which $|x - 1| + |y - 1| \leqslant 1$.

The STEP question

3 Sketch the following subsets of the x-y plane:

(i) $|x| + |y| \leqslant 1$;

(ii) $|x - 1| + |y - 1| \leqslant 1$;

(iii) $|x - 1| - |y + 1| \leqslant 1$;

(iv) $|x|\,|y - 2| \leqslant 1$.

This STEP question is not particularly easy; you have to be careful.

It is probably easiest to start each part by sketching the boundaries (given by replacing the inequality signs by equals signs) first. Then all you have to do is test one point in each region.

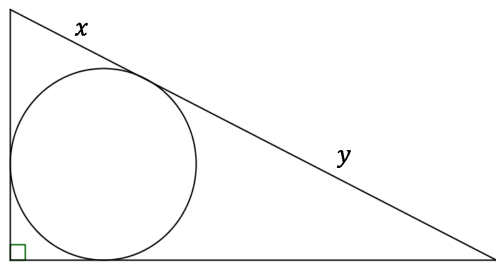
Note that the required region in part (ii) is just a translation of the required region in part (i). You could use this idea in parts (iii) and (iv) if you wanted. (You could have used the idea to obtain (ii) directly from (i).)

Discussion

In A-Level (or similar) examinations, you might be given a function $\mathrm{f}(x)$ and be asked to sketch $y = |\mathrm{f}(x)|$. This can give interesting-looking graphs but essentially all you have to do is take any parts of the graph that lie underneath the x-axis and reflect them about the x-axis. STEP questions are usually more complicated, and require a deeper understanding of the modulus function.

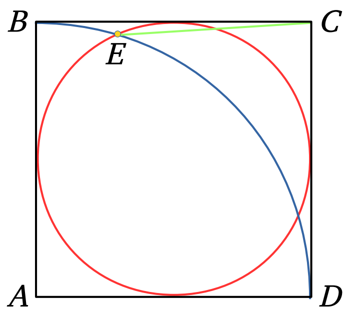
Warm down

4 **(i)** In the diagram below, the circle touches all three sides of the triangle. Find the area of the triangle in terms of x and y.



It will help to draw some radii onto the diagram.

(ii) In the diagram below, $ABCD$ is a square. The arc BD has centre A and radius AB. It meets the inscribed circle at point E. Show that the length CE is half the length of the diagonal of the square.



You can do this using pure geometry, by means of similar triangles (good!) or using trigonometry (goodish!) or using coordinate geometry (a bit laborious for this question!).

It would be a good idea to mark the centre of the square (call it O) on your diagram. You might then like to think about the various triangles with vertices chosen from A, C, O and E.