STEP CORRESPONDENCE PROJECT

Assignment 14

Warm-up

You probably already know that for a graph with gradient $\frac{dy}{dx}$:

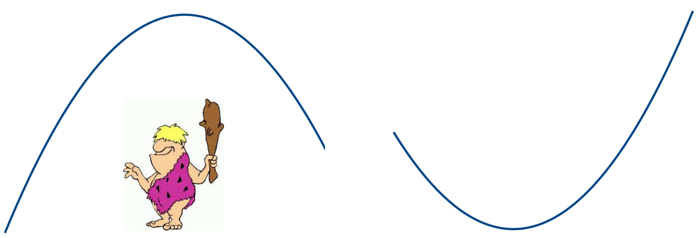
- if $\frac{dy}{dx} > 0$ then the graph is *increasing*;
- if $\frac{dy}{dx} < 0$ then the graph is *decreasing*.

The sign of the second derivative also tells us something about the shape of a graph:

- if $\frac{d^2y}{dx^2} < 0$ then the graph is *concave*, which means that the gradient of the curve is decreasing — it is bending downwards;
- if $\frac{d^2y}{dx^2} > 0$ then the graph is *convex*, which means that the gradient of the curve is increasing — it is bending upwards.

A convex curve is like a smile.

The picture below shows the difference. If you get muddled up, remember that a **cave**man can live in a con**cave** graph.



Concave
Gradient is decreasing

Convex
Gradient is increasing

A *point of inflection* is a point at which the graph changes from being concave to being convex or vice versa. At the point of inflection $\frac{d^2y}{dx^2} = 0$; in addition, the gradient must have the same sign on either of the point. Points of inflection can be stationary (if $\frac{dy}{dx} = 0$ as well) or non-stationary.

1 **(i)** Find a range of values of x for which the graph $y = x^4 - 6x^2 + 9$ is concave.

(ii) For the graph $y = x^3 - 2x^2 - 3x$, find the point where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$.

(iii) For the graph $y = x^4 - 2x^3$, find the stationary points and the points where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$. Sketch $y = x^4 - 2x^3$.

(iv) For the graph $y = (x-1)^4$, find the point where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$. By considering the shape of the graph show that this is not a point of inflection.

Note that **if** we have a point of inflection **then** $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at this point, but **if** $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$ at some point it does not necessarily mean that this point is a point of inflection.

Discussion

We stated that

- if $\dfrac{\mathrm{d}y}{\mathrm{d}x} > 0$ then the graph is *increasing*

and this is correct. *However*, it is not the definition of an increasing graph; it is a sufficient condition for the graph to be increasing, but not a necessary condition. In other words, it is 'if ... then' but not 'if and only if ... '.

The usual mathematical definition is:

- the graph $y = \mathrm{f}(x)$ is *increasing* if and only if $\mathrm{f}(x+h) \geqslant \mathrm{f}(x)$ for $h > 0$.

Note that the inequality is not strict, so the graph $y = 1$ is increasing according to this definition. This seems peculiar, but it turns out to be convenient.[1]

If we replace $\geqslant$ with $>$, we say that the graph is *strictly increasing*.

Note also that this definition works even for functions (such as flight of steps, or the floor function) which do not have a derivative at all points.

[1]For example, a (differentiable) function is increasing if and only if $\frac{\mathrm{d}y}{\mathrm{d}x} \geqslant 0$, which is convenient. On the other hand, it is not that case that a (differentiable) function is strictly increasing if and only if $\frac{\mathrm{d}y}{\mathrm{d}x} > 0$; think of $y = x^3$, which is certainly strictly increasing (it gets bigger and bigger!), but $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ (so is not strictly positive) at $x = 0$.

Preparation

2 **(i)** Consider the graph $y = x^3 - 3x + 2$. Sketch the graph by:

(a) factorising y and hence finding the roots, **and**
(b) finding the coordinates of the turning points, **and**
(c) finding where the curve crosses the y-axis.

How many distinct roots of the equation $x^3 - 3x + 2 = 0$ are there?

Note that the graph has a "repeated" root at $x = 1$. This occurs when a turning point is located on the x-axis. When asked for *distinct* roots, we count a repeated root only once.

(ii) By considering each of the following graphs as a transformation of $y = x^3 - 3x + 2$, sketch it (showing the coordinates of the turning points and y-intercept) and state how many distinct roots there are.

Note that you are not asked to find the *values* of the roots.

(a) $y = x^3 - 3x$
(b) $y = x^3 - 3x + 4$
(c) $y = x^3 - 3x - 4$

State the values of k for which the equation $x^3 - 3x + k = 0$ has **(A)** 2 distinct roots and **(B)** 3 distinct roots.

Use the idea of translating the original graph to help you. For part **(B)** there is a range of values of k.

(iii) Consider the graph of $y = 3x^4 + 4x^3 - 6x^2 - 12x + 5$.

(a) Find the coordinates of the stationary points.

(b) Find the points where $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2} = 0$. State whether each one is stationary or non-stationary.

(c) Sketch the graph.

(d) Write down the value of k for which the equation $3x^4 + 4x^3 - 6x^2 - 12x + k = 0$ has only one root.

The STEP question

3 **(i)** Sketch the curve $y = x^4 - 6x^2 + 9$ giving the coordinates of the stationary points.

Let n be the number of distinct real values of x for which

$$x^4 - 6x^2 + b = 0.$$

State the values of b, if any, for which (a) $n = 0$; (b) $n = 1$; (c) $n = 2$; (d) $n = 3$; (e) $n = 4$.

(ii) For which values of a does the curve $y = x^4 - 6x^2 + ax + b$ have a point at which both $\dfrac{dy}{dx} = 0$ and $\dfrac{d^2y}{dx^2} = 0$?

For these values of a, find the number of distinct real values of x for which

$$x^4 - 6x^2 + ax + b = 0\,,$$

in the different cases that arise according to the value of b.

(iii) Sketch the curve $y = x^4 - 6x^2 + ax$ in the case $a > 8$.

For part **(ii)** you could sketch $y = x^4 - 6x^2 + ax$ for the particular values of a.

This is a Paper II question; having just written out the solution, we think it is pretty long even for Paper II!

Warm down

I have a £1 coins and b £5 and c £10 notes (at least one of each). As it happens, the total amount of money would be unchanged if instead I had b £1, c £5 notes, and a £10 notes. I want to investigate the possible values of (a, b, c).

Consider first the case $a = 6$.

(a) There is one very obvious solution. What is it?

(b) Write down an equation relating b and c and sketch the graph of this equation in the b-c plane (i.e. with b and c on the axes).

(c) Show that if $(6, b, c)$ is a solution, then $(6, b + 5k, c - 4k)$, where k is an integer, is another solution. Is every solution of this form?

(d) Find the three solutions.

How is the situation different if $b = 6$ (instead of $a = 6$)?