STEP CORRESPONDENCE PROJECT

Assignment 15

Warm-up

1 **(i)** Show that $(a-b)(a^2+ab+b^2) = (a^3-b^3)$.

Find a similar factorisation for a^3+b^3.

These results are called the "difference of two cubes" (like the difference of two squares) and "the sum of two cubes" factorisations. They are useful, and you should be able to reproduce them without thinking too much.

(ii) Factorise **(a)** x^3+125 and **(b)** x^6-y^6.

(iii) Find the sum of the the following geometric progression:

$$1+t+t^2+t^3+t^4\,,$$

and deduce (by choosing t suitably) that $a^5-b^5 = (a-b)(a^4+a^3b+a^2b^2+ab^3+b^4)$ and write down a factorisation for a^5+b^5.

(iv) Simplify $(x^2+y^2-\sqrt{2}\,xy)(x^2+y^2+\sqrt{2}\,xy)$.

Use this result to find the four values of x that satisfy $x^4=-1$. (Your answers will involve $\sqrt{-2}$ which you can write as $\sqrt{2}\,i$ if you like.)

Preparation

2 **(i)** Simplify:

$$\frac{1}{3+\sqrt{5}}+\frac{1}{3-\sqrt{5}}\,.$$

(ii) Explain why the following infinite sum exists (i.e. converges).

$$1+\frac{1+\sqrt{3}}{3}+\left(\frac{1+\sqrt{3}}{3}\right)^2+\cdots$$

Evaluate the sum.

(iii) Expand $(2+\sqrt{3})^5$. Hence write down the expansion of $(2-\sqrt{3})^5$.

(iv) Solve the following simultaneous equations:

$$x^2 - y^2 = z$$
$$x - y = z$$
$$xy = -2$$

You may like to start by factorising $x^2 - y^2$.

The STEP question

3 The first four terms of a sequence are given by $F_0 = 0$, $F_1 = 1$, $F_2 = 1$ and $F_3 = 2$. The general term is given by

$$F_n = a\lambda^n + b\mu^n \,, \qquad (*)$$

where a, b, λ and μ are independent of n, and a is positive.

(i) Show that $\lambda^2 + \lambda\mu + \mu^2 = 2$, and find the values of λ, μ, a and b.

(ii) Use $(*)$ to evaluate F_6.

(iii) Evaluate $\displaystyle\sum_{n=0}^{\infty} \frac{F_n}{2^{n+1}}$.

Discussion

You may have come across the sequence F_0, F_1, F_2, ... determined by

$$F_n = F_{n-1} + F_{n-2} \,.$$

This is called the *Fibonacci sequence*, after a 12th century scholar who lived in Pisa, Italy who was interested in rabbits. The recurrence relation can be solved to obtain the general term, which is used in this question.

The terms of the Fibonacci sequence have remarkable properties, which have provided a rich seam for STEP questions over the years.

Warm down

4 The Reverend Thomas Kirkman posed a problem in the 1850 edition of *The Lady's and Gentleman's Diary*[1], which became famous. In fact, it became the inspiration for a whole area of mathematics (part of Combinatorics).

Kirkman wrote:

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast[2].

This is very topical, because it was only very recently (2015) that a mathematician from Oxford solved the outstanding question of whether this kind of problem has, in general, any solutions. The answer is yes, subject to some simple conditions.

Your task is to try to solve the 9-schoolgirl, rows of three, four day, problem. The answer is unique, though of course the young ladies could change their names to obtain a seemingly different but mathematically equivalent solution. I suggest representing the first day's walk in the form ABC, DEF, GHI.

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[1] This publication contained mathematical problems for ladies and gentlemen to solve; those were the days!

[2] i.e. they were arranged in a 3×5 rectangle. The 'abreast' condition is a neat way of saying that if Alice and Beryl (say) walk in the same triplet on one day, then they must walk in different triplets on all the other days. It is not clear why they had to be ladies.