STEP CORRESPONDENCE PROJECT

Assignment 17

Warm-up

1 **(i)** The function f satisfies

$$f(x^2) = \sqrt{1 + x^4}\,.$$

Write down an expression for f(y). On what domain (i.e. for what range of values of y) does your expression hold?

(ii) The function f satisfies

$$f(\sqrt{x}) = \sqrt{1 + x^4}\,.$$

Write down an expression for f(y). On what domain (i.e. for what range of values of y) does your expression hold?

(iii) The function f satisfies

$$f'(x) = 3x^2 \quad \text{and} \quad f(0) = 0\,.$$

Write down an expression for f(x).

Note: f′ is defined by $f'(x) = \dfrac{df}{dx}$.

(iv) The function f satisfies

$$f'(x^2) = 3x^4 \quad \text{and} \quad f(0) = 0\,.$$

Write down an expression for f(y).

(v) The function f satisfies

$$f(x + y) = f(x)f(y) \quad \text{and} \quad f(1) = 2\,.$$

Verify that $f(x) = a^x$ is a possible solution, for a suitably chosen constant a.

Note: 'verify' means 'check that it works' — start with the given solution rather than trying to derive it (which would be tricky).

(vi) The function f satisfies

$$f(xy) = f(x) + f(y) \quad \text{and} \quad f(2) = 1\,.$$

Can you write down a possible solution? (Again, don't try to *derive* it).

(vii) The function f satisfies

$$\mathrm{f}(x+y) = \mathrm{f}(x) + \mathrm{f}(y) \quad \text{and} \quad \mathrm{f}(2) = 1\,.$$

Can you write down a possible solution?

Preparation

2 **(i)** In Assignment 10 you obtained the following results:

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta \quad \text{and} \quad \cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

Note the use of $\mp$ for the cosine, meaning that when there is $+$ on the left hand side, there is $-$ on the right hand side (and vice-versa).

Use these to find $\sin 105°$ and $\cos 75°$.

(ii) By first writing $\cos 3A = \cos(2A + A)$, write $\cos 3A$ in terms of $\cos A$. Use this to solve the equation $\cos 3A + \cos A = 2\cos 2A$, for $0 \leqslant A < 2\pi$.

The identities for $\sin 2A$ and $\cos 2A$ will probably be helpful. You can derive these by substituting $\alpha = \beta = A$ into the results in part **(i)**. The identity $\sin^2 x + \cos^2 x = 1$ will probably be useful as well.

(iii) Show that $1 + \sqrt{2}$ is a root of the equation:

$$2x^3 - (2\sqrt{2}+6)x^2 + (4\sqrt{2}+5)x - \sqrt{2} - 1 = 0\,.$$

Hence find the other two solutions.

To show that $1+\sqrt{2}$ is a root, substitute $x = 1+\sqrt{2}$ into the equation. You will have to expand carefully various brackets, but it would be a good idea to look carefully at your expression before starting to expand the brackets in case any simplification is possible.

To find the other roots you can divide by $(x - 1 - \sqrt{2})$ to obtain a quadratic; again care is needed but the result turns out to be rather simple.

(iv) Solve the equation $2x^3 - 5x^2 - 6x + 9 = 0$. Use this equation, and a substitution of the form $x = ky$ (for a suitably chosen k), to obtain the solutions of the equation $6y^3 - 5y^2 - 2y + 1 = 0$.

The STEP question

3 **(i)** Show that $\cos 15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}}$ and find a similar expression for $\sin 15^\circ$.

(ii) Show that $\cos\alpha$ is a root of the equation

$$4x^3 - 3x - \cos 3\alpha = 0\,,$$

and find the other two roots in terms of $\cos\alpha$ and $\sin\alpha$.

(iii) Use parts (i) and (ii) to solve the equation $y^3 - 3y - \sqrt{2} = 0$, giving your answers in surd form.

Discussion

The Indian mathematician Brahmagupta (597–668 AD) was the first to obtain the formula for solving quadratic equations. Since algebra (the use of letters in equations) did not exist at that time, he had to write his discovery in words: *To the absolute number multiplied by four times the coefficient of the square, add the square of the coefficient of the middle term; the square root of the same, less the coefficient of the middle term, being divided by twice the coefficient of the square is the value.*

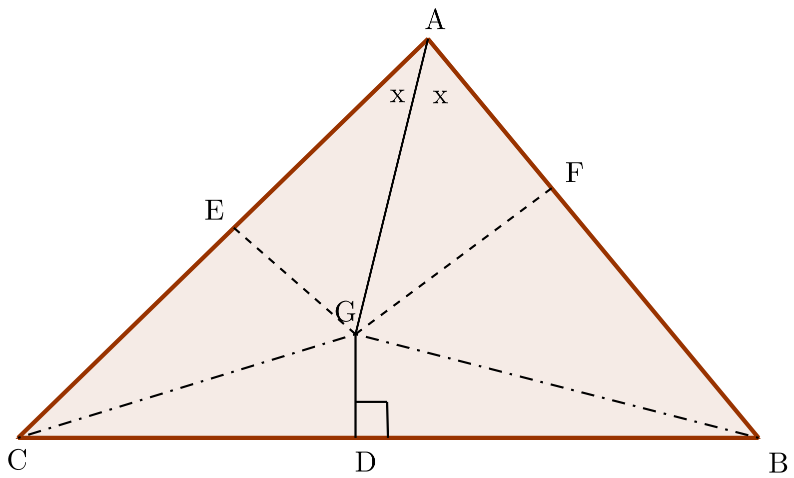
No such formula for solving the general cubic was discovered for another 1000 years; and then another 300 years elapsed before it was proved that no formula for the general quintic could exist.

One method of solving the cubic involved trigonometric identities, and the STEP question above is an example of this method, though this example works only in special cases.

Warm down

4 The diagram below shows (not very accurately) a triangle ABC, in which the bisector of angle A meets the perpendicular bisector of the line BC at the point G. (The point D is the midpoint of BC).

The lines GE and GF are the perpendiculars from G to AC and AF.



(i) By considering the triangles AGE and AGF, show that $AE = AF$.

(ii) Show that $GC = GB$.

(iii) Show that $EC = FB$ and deduce that triangle ABC is isosceles.

(iv) How do you account for this? (Hint: draw a more accurate diagram.)