

STEP CORRESPONDENCE PROJECT**Assignment 19****Warm-up**

- 1 When sketching curves, the things to consider are intercepts with the x and y axes, turning points and asymptotes (i.e. what happens as $x \rightarrow \pm\infty$ or $y \rightarrow \pm\infty$). Often, you can sketch the curve without using all this information (though it is good to have it all). You may find it useful to think about transformations (translations in particular).

(i) Sketch the curve $y = \frac{1}{x-1}$.

(ii) Sketch the curve $y = \frac{x}{x-1}$.

What happens as $x \rightarrow \pm\infty$? It may be helpful to write $\frac{x}{x-1}$ in the form $a + \frac{b}{x-1}$.

(iii) Sketch the curve $y = \frac{x^2}{x-1}$.

Here you can divide x^2 by $x-1$ to enable you to write $y = \frac{x^2}{x-1}$ in the form $y = ax + b + \frac{c}{x-1}$. You can then see that as $x \rightarrow \pm\infty$ then $y \approx ax + b$. Be careful with your long division and double check that your answer still gives $y = \frac{x^2}{x-1}$.

You may need the derivative of $(x-1)^{-1}$ which is $-(x-1)^{-2}$, by the chain rule.

(iv) Sketch the curve $y = \frac{1}{x-1} + \frac{1}{x+1}$.

A small check on the general shape is to see whether the curve approaches the horizontal axis (as $x \rightarrow \pm\infty$) from above or from below. It may be helpful to consider the sign of the gradient.

Preparation

- 2 (i) Solve the quadratic equation $(a + 2)x^2 - 2x - a = 0$.
- (ii) For what values of x is $-7x - 7 < 0$? Show that $3x^3 - x^2 - 7x - 7 < 0$ when $x = -\frac{7}{9}$.
You don't have to calculate $(-\frac{7}{9})^3$.
- (iii) If $a/b > 0$, what can be said about a and b ?
Let $y = \frac{x}{x-1}$.
If $y > 0$ and $x > 0$, show that $x > 1$. If in addition $y > x$, show that $x < 2$.
You could use the sketch from Q1(ii), but it is better from the point of view of the STEP question to do it without a sketch: the argument is just a few lines.

The STEP question

- 3 The numbers a and b , where $b > a \geq 0$, are such that

$$\int_a^b x^2 dx = \left(\int_a^b x dx \right)^2.$$

- (i) In the case $a = 0$ and $b > 0$, find the value of b .
- (ii) In the case $a = 1$, show that b satisfies

$$3b^3 - b^2 - 7b - 7 = 0.$$

Show further, with the help of a sketch, that there is only one (real) value of b that satisfies this equation and that it lies between 2 and 3.

- (iii) Show that $3p^2 + q^2 = 3p^2q$, where $p = b + a$ and $q = b - a$, and express p^2 in terms of q . Deduce that $1 < b - a \leq \frac{4}{3}$.

Discussion

In part (ii) you may get a quartic in b . If so, can you spot an “obvious” root?

In part (iii) you have to return to the integrals given in the very first line of the question and try to write the expression you get in terms of p and q .

Warm-down

4 Here is the set-up for this question. You should justify your answers as *briefly* as possible.

- We have an island with 100 islanders living on it.
 - Islanders have either blue eyes or brown eyes.
 - Islanders meet at 9.00 each morning and look into each others' eyes.
 - Islanders never discuss eye-colour. There are no mirrors or cameras on the island.
 - Any islander who finds out that he or she has blue eyes must leave the island by the 12.00 boat.
 - We join the islanders, in spirit anyway, on day one when a ghostly voice at the start of the 9.00 meeting says '*At least one of you has blue eyes*'.
- (i) Suppose that no islander leaves the island on the 12.00 boat on day one. What can you conclude?
- (ii) Suppose that one islander leaves the island on the 12.00 boat on day one. What can you conclude?
- (iii) Suppose instead that no one leaves on the first day, but two islanders leave on the 12.00 boat on the second day. What can you conclude?
- (iv) Suppose instead that we don't know if anyone leaves on the first day, but we do know that two islanders leave on the 12.00 boat on the second day. What can you conclude?
- (v) Suppose that n islanders leave on the n th day. What can you conclude?