# STEP CORRESPONDENCE PROJECT

## Assignment 1

## Warm-up

**1** **(i)** Simplify $(2x-3)^2-(x-1)^2$. Check your answer by evaluating it for $x=1$ and $x=2$.

**(ii)** Show that

$$\sqrt{1+x^2}\ -x=\frac{1}{\sqrt{1+x^2}\ +x}\,.$$

Deduce that if $x$ is very large, then $\sqrt{1+x^2}-x$ is approximately equal to $\dfrac{1}{2x}$.

## Preparation

**2** **(i)** Sketch the line $y=x+1$ for $-2\leqslant x\leqslant 2$.

What is the greatest value of $x+1$ in this range?

**(ii)** Sketch the curve (parabola) $y=(x-1)^2$ for $-2\leqslant x\leqslant 2$.

What is the greatest value of $(x-1)^2$ in this range?

**(iii)** What are the greatest and least values of $(x-3)^2$ for $-2\leqslant x\leqslant 2$?

**(iv)** Write the expression $x^2-8x+21$ in the form $(x+a)^2+b$. Hence find the coordinates of the minimum point of the curve $y=x^2-8x+21$.

**(v)** Sketch the curve (parabola) $y=x^2+2kx$ for $-2\leqslant x\leqslant 2$, where $-2<k<2$.

What are the greatest and least values of $y=x^2+2kx$ for $-2\leqslant x\leqslant 2$?

What would your answers be if $k>2$?

## The STEP question

**3** **(i)** Find the greatest and least values of $bx + a$ for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases $b > 0$, $b = 0$ and $b < 0$.

**(ii)** Find the greatest and least values of $cx^2 + bx + a$, where $c \geqslant 0$, for $-10 \leqslant x \leqslant 10$, distinguishing carefully between the cases that can arise for different values of $b$ and $c$.

## Postmortem

This question has some features that are very typical of STEP questions.

There is a first part, which is fairly straightforward (not necessarily *easy*; but you know what you have to do). But then there is a second part, similar in some respects to the first part, in which you have to do quite a lot of thinking.

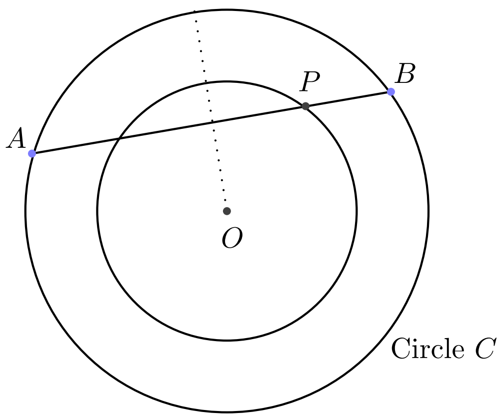
Also, you have some numbers that are respresented by letters ($a$, $b$ and $c$) — the correct term for these letters is *parameters* — and you have to decide for yourself how the result you are trying to prove depends on these parameters.

Finally, although you are not told to draw sketches, you should do so: it is much easier to work on the different cases if you have sketches in front of you.

## Warm down

**4** The diagram shows a circle, $C$, with centre $O$ and a rod $AB$ the ends of which can slide round the circle $C$ (so that $AB$ is a chord of $C$). The radius of the circle is $R$ and the length the rod is $2a$.

As the rod slides round $C$, the point $P$ which is a fixed distance $b$ from the centre of the rod, traces out a circle with centre $O$ of radius $r$.




Show that the area between the two circles is $\pi(a^2 - b^2)$.

The surprising thing about this result is that it is independent of the radius of $C$ (assuming that it is greater than $a$) and depends only on the length of the stick and the position of $P$ on the stick. It doesn't matter how big the circle is, the area between the two circles is always $\pi(a^2 - b^2)$.

If you had known that the answer was independent of $R$, can you think of an easy way (by choosing $R$) of obtaining the result?

Even more surprising is the fact that the result holds even when $C$ is not a circle, but is any closed curve round which the rod can slide smoothly. This is Holditch's theorem, proved in about 1840, and not much seen until it was used as a STEP question in 2010.

The lengths $R$ and $r$ are given here for your convenience: they are required for the calculation but do not appear in the answer. In a STEP question, it would have been up to you to decide what is needed for the calculation. And you might have had to draw the diagram yourself.