STEP CORRESPONDENCE PROJECT

Assignment 2

Warm-up

1 **(i)** Expand and simplify

$$(x - y + z)^2 \,.$$

(ii) Simplify

$$\frac{x}{x^2 - y^2} - \frac{y}{(x-y)^2} - \frac{1}{x+y} \,.$$

(iii) Find all real values of x that satisfy

$$x^2 - 1 = \frac{1}{x^2} \,.$$

Leave your answers in the form of surds.

Preparation

2 **(i)** Solve the equation:

$$\frac{2}{x+3} + \frac{1}{x+1} = 1 \,.$$

(ii) Find the value(s) of b for which the following equation has a single (repeated) root.

$$9x^2 + bx + 4 = 0 \,.$$

The use of the discriminant of a quadratic equation to determine the nature of the roots is a fairly basic tool that you will be expected to use fluently.

(iii) Find the range of (real) values of c for which the following equation has no real roots:

$$3x^2 + 5cx + c = 0 \,.$$

The STEP question

3 In this question a and b are distinct, non-zero real numbers, and c is a real number.

(i) Show that, if a and b are either both positive or both negative, then the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1$$

has two distinct real solutions.

(ii) Show that, if $c \neq 1$, the equation

$$\frac{x}{x-a} + \frac{x}{x-b} = 1 + c$$

has exactly one real solution if

$$c^2 = -\frac{4ab}{(a-b)^2} \, .$$

Show that this condition can be written

$$c^2 = 1 - \left(\frac{a+b}{a-b}\right)^2$$

and deduce that it can only hold if $0 < c^2 \leqslant 1$.

Postmortem

This question is relatively straightforward, but it does require fluent algebraic manipulation and a clear head. To gain maximum credit your layout needs to be clear — the mantra "One equal sign per line, all equal signs aligned." is a good one to follow. Where the answer is given ("Show that") you must show sufficient working to justify the given answer; any missing steps will be penalised. Note that the last part uses the work "deduce" — this means that you must use the previous part and also implies that little further working is required; however your explanations must still be clear and logical.

Warm down

4 A Minister and a Bishop were having a cup of tea. There was a knock at the door, and three bell ringers entered the room. After introductions, the Bishop asked the Minister how old the bell ringers were.

"Well," the Minister said, knowing the Bishop had a penchant for numerical puzzles, "if you multiplied their three ages together, you'd get 2,450. But if you added them, you'd get twice your age."

"Hmm," the Bishop muttered, after several moments' thought. "I haven't enough information to solve that."

"It may help, my dear Bishop," offered the Minister, "to know that I am older than anyone else here in the room."

"Yes, indeed it would," replied the Bishop. "Now I know their ages."

The question is: How old is the Minister?

You may assume that all ages are integers.

This puzzle is intended mainly for amusement, but also to initiate a quick discussion about the peculiarly English pursuit of Change Ringing. The bells are numbered from lightest (and highest pitched) to heaviest such as 1, 2, 3, 4, 5, 6. The lightest bell is called the Treble and the heaviest the Tenor. When they ring in descending order it is known as "rounds". When a "method" is rung, the bells must not repeat any permutation of the bells so, for example, the first two rows of "Plain Bob Minor" are (2 1 4 3 6 5) and (2 4 1 6 3 5). If any permutation is repeated the method is "false".

There are further rules to follow in that any bell can only change position by one each stroke, and usually speaking each bell (apart from the Treble) follows the same pattern throughout the method.

If all the possible permutations of the bells method are rung (once and once only!) then this is known as an "extent" or "full peel". Usually a method will not give all the possible permutations and the method is varied at key moments (when the Treble is at the front) by calling a "bob" or "single". The length of an extent on 5 bells is $5! = 120$ and takes about 5 minutes to ring. An extent on 8 bells has only ever been rung once which was in 1963 in Loughborough: it took just under 18 hours. If an extent were rung on the 12 bells of St Mary's in Cambridge it would take over 30 years.