STEP CORRESPONDENCE PROJECT

Assignment 23

Warm-up

1 This question is all about the chain rule. If you already know the chain rule, please forget it for the moment.

(i) Let $\mathrm{g}(x) = 2x^3 + 1$ and $\mathrm{f}(x) = x^2$. Let $\mathrm{F}(x) = \mathrm{f}\big(\mathrm{g}(x)\big)$. Find an expression for $\mathrm{F}(x)$ in terms of x. Find $\mathrm{F}'(x)$ and show that $\mathrm{F}'(x) \neq \mathrm{f}'\big(\mathrm{g}(x)\big)$.

(ii) Use the formulae

$$\sin(A+B) = \sin A\cos B + \cos A \sin B \quad \text{and} \quad \cos(A+B) = \cos A\cos B - \sin A\sin B$$

to obtain expressions for $\sin 2\alpha$ and $\cos 2\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$.

Let $\mathrm{f}(x) = \sin x$ and $\mathrm{g}(x) = 2x$. Let $\mathrm{F}(x) = \mathrm{f}\big(\mathrm{g}(x)\big)$. Find an expression for $\mathrm{F}(x)$ in terms of x. Using the rule for differentiating a product (and without using the chain rule!), show that $\mathrm{F}'(x) = 2\cos(2x)$.

(iii) Now we assume that for any differentiable[1] function H, the approximation

$$\mathrm{H}(a+t) \approx \mathrm{H}(a) + t\mathrm{H}'(a)$$

is valid when t is small.

(a) Write down an approximation for $\mathrm{g}(x+h)$, where g is a differentiable function and h is small.

(b) Write down an approximation for $\mathrm{f}\big(\mathrm{g}(x)+t\big)$, where f is a differentiable function and t is small.

(c) Let $\mathrm{F}(x) = \mathrm{f}\big(\mathrm{g}(x)\big)$. Use your answers to (a) and (b) to obtain an approximation for $\mathrm{F}(x+h)$ when h is small.

(d) Deduce that that $\mathrm{F}'(x) = \mathrm{f}'\big(\mathrm{g}(x)\big)\mathrm{g}'(x)$. This is the chain rule for differentiating a 'function of a function', i.e. the composition of two functions.[2]

[1] You mustn't worry about what 'differentiable' means. But to give an example, the function $|x|$ is: continuous everywhere, because you can draw it without taking the pen off the paper; differentiable everywhere except $x = 0$, because its gradient is just ± 1; but not differentiable at $x = 0$ because it is 'pointed' and its gradient can't be calculated.

[2] As in previous assignments, this proof is not perfect: we have to worry about what 'small' means, and whether the error in the approximation can matter when we take the limit. This proof can be made rigorous but, as before, we leave proper consideration of these things to your lecturers when you get to university.

(iv) Use the chain rule to find the derivative with respect to x of $(2x^3+1)^2$ and $\sin(2x)$.

(v) Differentiate $\ln(2x)$ and $\ln(x^2)$ by two different methods, first using the chain rule and second and without using the chain rule.[3]

Preparation

2 **(i)** Expand $\left(5-\frac{4}{3}y-2t\right)^2$.

(ii) Solve the following sets of equations. In each case, sketch the two curves, and indicate the point(s) if intersection if any.

(a) $y=3x-1$ and $(x-4)^2+(y-1)^2=9$

(b) $y=2x-3$ and $(x-5)^2+(y-2)^2=5$

(c) $3y=x+5$ and $x^2+y^2-6x-2y-15=0$

(iii) The straight line L has equation $y=ax-1$ and the curve[4] C has equation $y=x^2$.

For what range of values of a does L intersect C in two distinct points?

Find the values of a for which L touches C (i.e. meets C without crossing it). Hence write down the equations of the two tangents to C that pass through the point $(0,-1)$.

(iv) Use the formulae for $\sin 2\alpha$ and $\cos 2\alpha$ in terms of $\sin\alpha$ and $\cos\alpha$ to show that

$$\tan 2\alpha = \frac{2\tan\alpha}{1-\tan^2\alpha}\,.$$

[3]Recall that $\ln x = \log_e x$ and that $\dfrac{d}{dx}\ln x = \dfrac{1}{x}$.

[4]It is a parabola.

The STEP question

3 **(i)** The equation of the circle C is

$$(x - 2t)^2 + (y - t)^2 = t^2,$$

where t is a positive number. Show that C touches the line $y = 0$.

Let α be the acute angle between the x-axis and the line joining the origin to the centre of C. Show that $\tan 2\alpha = \frac{4}{3}$ and deduce that C touches the line $3y = 4x$.

(ii) Find the equation of the incircle of the triangle formed by the lines $y = 0$, $3y = 4x$ and $4y + 3x = 15$.

Note: The *incircle* of a triangle is the circle, lying totally inside the triangle, that touches all three sides.

Discussion

A picture is worth a thousand words.[5] You can do much of part (i) using sketches for which little explanation is needed.

Warning: there is some algebra in part (ii), with big numbers. For the '$b^2 = 4ac$' condition required for the circle to touch $4y + 3x = 15$, you can do some cancellation to get $(t - 4)^2 = (4t^2 - 20t + 25)$.

You then find two possible values of t, and again a sketch would be sufficient to show which is the required value.

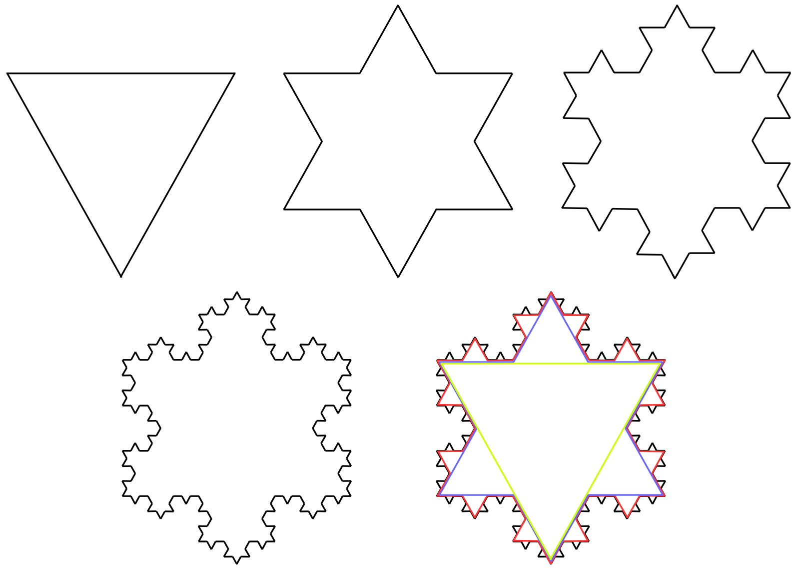
[5] This quote is sometimes attributed to Napoleon Bonaparte, who said 'Un bon croquis vaut mieux qu'un long discours' or 'A good sketch is better than a long speech'.

Warm down

4 This question is about a *fractal* curve.[6,7] The diagrams below show how our fractal is created. We start with an equilateral triangle. Then each side is divided into three equal parts and the middle section is replaced by two sides of an equilateral triangle. There are now 12 edges.

At the next step each of these edges is divided into three equal parts and again the middle section is replaced by two sides of an equilateral triangle.

The process is then repeated over and over again. The first four diagrams below show the original triangle and the first three iterations. The final diagram just shows these iterations superposed.



(i) How many edges are there in the second iteration? Find an expression for the number of edges in the n^{th} iteration.

(ii) Let each side of the original triangle have length 1. How long is each of the 12 edges in the first iteration? Write down an expression for the length of each edge in the n^{th} iteration.

(iii) Using your answers to parts **(i)** and **(ii)** find an expression for the total length of the curve in the n^{th} iteration. What happens to this total length as $n \to \infty$?

(iv) Let the area of the original triangle be A. By considering the areas of each of the 3 small triangles added in the first iteration, find the total area of the first iteration of the curve. Similarly, find an expression for the total area of the second iteration.

Obtain an expression for the total area of the n^{th} iteration as a sum of the form $A + \frac{1}{3}A(1 + r + r^2 + \cdots + r^{n-1})$ for some number r which you should find. What happens to the total area as $n \to \infty$?

[6] Fractals are infinitely complex patterns that are self-similar across different scales. They are created by repeating a simple process over and over.

[7] You may have heard Elsa singing about "Frozen fractals all around".