STEP CORRESPONDENCE PROJECT

Assignment 25

Warm-up

1 This assignment is mainly concerned with integration by substitution.

In general, substitution works like this. We are trying to integrate $f(x)$, without success. So we replace x with some function $g(u)$ that we hope will make the integral easier:

$$\int f(x)\,dx = \int f(g(u)) \times \frac{dx}{du}\,du \qquad (*)$$

where $x = g(u)$. Here, $\dfrac{dx}{du}$ means $\dfrac{dg(u)}{du}$. We have done two things: we have replaced every x in the integrand with $g(u)$, as we said we would, and we have also replaced dx with $g'(u)\,du$ (it is more memorable, but a little mathematically uncouth, to have in mind '$dx = \frac{dx}{du}du$'). If the integral had limits, there would be a third thing to do (see below).

You may have covered this in your A-level (or IB, etc) studies. If you have not, this assignment will take you a little longer, but keep returning to $(*)$ and it should be fine.

(i) *Indefinite integrals*

(a) $\displaystyle\int \frac{x}{x-2}\,dx$ (substitute $x = u + 2$) .

(b) $\displaystyle\int \frac{6x}{\sqrt{2x+1}}\,dx$ (substitute $x = \frac{1}{2}(u^2 - 1)$) .

Once you have everything in terms of u and have integrated, you must remember to rewrite anything in terms of u in terms of x; to do this, you will need to express u in terms of x.

(ii) *A definite integral*

Use the substitution $x = 2\sin\theta$ to evaluate $\displaystyle\int_0^1 \frac{1}{\sqrt{4-x^2}}\,dx$.

We have used θ instead of u, which makes no difference: there is nothing special about u — you can call the new variable anything you want (except x).

One way of doing the definite integral is to treat it first as an indefinite integral and, at the very end of the calculation, evaluate at the upper and lower limits.

However, you can instead be a bit clever, and rewrite the integral with new limits corresponding the values taken by the new variable. Perhaps it is best to do the above integral this way, since that is what you need to do for the STEP question.

In this case, the lower limit $x = 0$ corresponds to $\sin\theta = 0$ i.e. $\theta = 0$.[1] The upper limit is a little more interesting).

[1] There are lots of values of θ that give $\sin\theta = 0$, but we can choose to use the "principal value", which in the case of $\sin\theta$ is the one that lies in the range $-\frac{\pi}{2} < \theta \leqslant \frac{\pi}{2}$. Note θ must be in radians. (Why?)

Preparation

2 **(i)** Starting from

$$\tan(A-B) = \frac{\sin(A-B)}{\cos(A-B)}\,,$$

find an expression for $\tan(A-B)$ in terms of $\tan A$ and $\tan B$.

Recall that $\sin(A+B) = \sin A\cos B + \cos A\sin B$; see Assignment 10 for more on this.

(ii) Simplify $\ln\left(1+\dfrac{\frac12 - x}{\frac12 + x}\right)$.

(iii) By starting with $\sin^2\theta + \cos^2\theta \equiv 1$, find a relationship between $\tan\theta$ and $\sec\theta$.

Note that $\sec\theta$ is defined by $\sec\theta = \dfrac{1}{\cos\theta}$.

(iv) Show that $\dfrac{\mathrm{d}(\tan\theta)}{\mathrm{d}\theta} = \sec^2\theta$.

You can use the product rule for this, if you write $\tan\theta$ as a product of $\sin x$ and $\sec x$; or the quotient rule if you prefer.

(v) Show that $\dfrac{1+\sin 2\alpha}{1+\cos 2\alpha} = \frac12(1+\tan\alpha)^2$.

Lots of ways of doing this; you could use $1 = \cos^2\alpha + \sin^2\alpha$ in the numerator and, if you like, in the denominator as well.

(vi) Let

$$I = \int_0^{84} \frac{x^2}{x^2+(84-x)^2}\,\mathrm{d}x\,. \qquad (\dagger)$$

Use the substitution $x = 84-u$ to obtain

$$I = \int_0^{84} \frac{(84-u)^2}{u^2+(84-u)^2}\,\mathrm{d}u\,.$$

Remember that in general $\int_a^b \mathrm{f}(x)\,\mathrm{d}x = -\int_b^a \mathrm{f}(x)\,\mathrm{d}x$

Now make the substitution $u = x$ in this integral and add it to $(\dagger)$ to show that $I = 42$.

The STEP question

3 By means of the change of variable $\theta = \frac{1}{4}\pi - \phi$, or otherwise, show that

$$\int_0^{\frac{1}{4}\pi} \ln(1 + \tan\theta)\, \mathrm{d}\theta = \tfrac{1}{8}\pi \ln 2.$$

Evaluate

$$\int_0^1 \frac{\ln(1+x)}{1+x^2}\, \mathrm{d}x \qquad \text{and} \qquad \int_0^{\frac{1}{2}\pi} \ln\left(\frac{1+\sin x}{1+\cos x}\right) \mathrm{d}x.$$

Discussion

Unlike A-Level questions, STEP questions do not tend to tell you what substitution to use. Since you have found the integral of $\ln(1+\tan\theta)$ in the first part, trying to get to something of this form might be a wise approach. You can also look at the limits to get an idea of what sort of substitution might be helpful.

Try some intelligent guesses first (remember this question is designed to take you 30 minutes or more!) and then, if you are still stuck, try the forums.

When you have finished the question, you might like to try the effect of the substitution $x = \frac{1}{2}\pi - u$ in the last integral.

Warm down

4 **(i)** Let

$$I = \int_0^a \frac{\mathrm{f}(x)}{\mathrm{f}(x) + \mathrm{f}(a-x)}\, \mathrm{d}x\,,$$

where $\mathrm{f}(x)$ is some unspecified function[2].

Show (remembering question 2(vi) of this assignment) that

$$I = \int_0^a \frac{\mathrm{f}(a-x)}{\mathrm{f}(x) + \mathrm{f}(a-x)}\, \mathrm{d}x$$

and deduce that $I = \frac{1}{2}a$.

Hence evaluate:

$$\int_0^{\frac{1}{2}\pi} \frac{\cos x}{\cos x + \sin x}\, \mathrm{d}x\,.$$

[2]It doesn't matter what it is, provided the denominator does not become zero for $0 \leqslant x \leqslant a$; that would be bad news.

(ii) If we differentiate $\ln(\mathrm{f}(x))$ using the chain rule, we get $\dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$. Therefore,

$$\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln(\mathrm{f}(x)) + c\,. \qquad (\ddagger)$$

We are assuming that $\mathrm{f}(x) > 0$ so that that the logarithm makes sense.[3]

(a) Use (‡) to find

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\cos x}{\sin x}\,\mathrm{d}x\,.$$

(b) By expressing the integrand in the form $\dfrac{\mathrm{f}'(x)}{\mathrm{f}(x)}$, find

$$\int \frac{1}{x\ln x}\,\mathrm{d}x\,.$$

(c) Suppose that

$$S = \int \frac{\cos x}{\cos x + \sin x}\,\mathrm{d}x \quad \text{and} \quad T = \int \frac{\sin x}{\cos x + \sin x}\,\mathrm{d}x.$$

By considering $S + T$ and $S - T$ determine S and T.

[3] In general $\int \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}\,\mathrm{d}x = \ln|\mathrm{f}(x)| + c$.