STEP CORRESPONDENCE PROJECT

Assignment 26

STEP I question

1 Preparation

(i) Find the equation of the tangent to the curve $y = \sqrt{x}$ at the point where $x = 4$.

Recall that $\sqrt{x}$ means the positive square root.

(ii) Solve the simultaneous equations $ax + by = c$ and $y = 2ax$, where $b \neq -\frac{1}{2}$.

(iii) Show that the equation of the normal to the curve $y = x^2$ at the point where $x = p$ is $2py + x = 2p^3 + p$.

(iv) Show that $(a - b)(a^2 + ab + b^2) = a^3 - b^3$. Simplify (for $a \neq b$):

$$\frac{a^3 - b^3}{a^2 - b^2}\,.$$

2 The STEP I question

The curve C has equation $xy = \frac{1}{2}$. The tangents to C at the distinct points $P\big(p, \frac{1}{2p}\big)$ and $Q\big(q, \frac{1}{2q}\big)$, where p and q are positive, intersect at T and the normals to C at these points intersect at N. Show that T is the point

$$\left(\frac{2pq}{p+q}\,,\,\frac{1}{p+q}\right).$$

In the case $pq = \frac{1}{2}$, find the coordinates of N. Show (in this case) that T and N lie on the line $y = x$ and are such that the product of their distances from the origin is constant.

Discussion

Note that your answers for the points of intersection should be symmetrical in p and q, i.e. you should get the same result when you switch p and q around. This is because you get the same point of intersection if you swap the points P and Q. This is a good check of your algebra.

Note also that if $p = q$ (which is not allowed in the question) then P and Q are the same point, the two tangents are concurrent (i.e. they are the same line), and every point on the tangent is a possible solution of the equations. This suggests that every term will have a factor of $p - q$ so that the equations are automatically satisfied when $p = q$. This factor can be divided out, though when you do so you should state 'p and q are distinct so $p - q \neq 0$'.

When finding the coordinates of N, you can choose to use $pq = \frac{1}{2}$ at different points in your argument. One approach is not to use it until you get to:

$$(p+q)x = p^2 + q^2 + pq + \frac{1}{4pq}\,.$$

If you write $p^2 + q^2 + pq = (p+q)^2 - pq$, it all comes out nicely in terms of $p + q$. Then you can find the corresponding y.

STEP III question

3 Preparation

The STEP question involves complex numbers and the Argand diagram. However, after you have put $z = x + \mathrm{i}y$ in the given equations and set the real and imaginary parts (separately) equal to zero, there are no more complex numbers left, and the Argand diagram is just the x-y plane.

In the first part, the locus given by the second equation (for which $y = 0$) is just the x-axis or part of it. Rather surprisingly, we are told to obtain p in terms of x, not x in terms of p. The question we are trying to answer is 'What values of x are allowed', in other words which values of x correspond to some value of p (it has to be a real value of p since that is what we were told and that is what we assumed when we took real and imaginary parts of the given equations). The equation

$$-p = x + \frac{1}{x}$$

shows that, for any value of x (except $x = 0$), there is a corresponding value of p. So the required locus is the whole of the x-axis except for $x = 0$.

It may be best to plunge straight into the STEP question without any preparation. But some of the sketches (especially for last part) are quite hard, so you might like to start with:

(i) Find the centre and radius of the circle $x^2 + y^2 + 2x = 0$.

(ii) For which values of x is $x^4 - x$ non-negative (i.e. greater than or equal to zero)?

For which values of x is $-x^2 - \dfrac{1}{x}$ non-negative?

(iii) Sketch the graphs $y = x - 1$, $y = \sqrt{x-1}$, and $y^2 = x - 1$, paying attention to the gradient of the second and third graphs at $x = 1$. Note that in the second graph $y \geqslant 0$, but in the third y can be negative (though we must have $x - 1 \geqslant 0$).

4 The STEP III question

(i) Let $x + \mathrm{i}y$ be a root of the quadratic equation $z^2 + pz + 1 = 0$, where p is a real number. Show that $x^2 - y^2 + px + 1 = 0$ and $(2x + p)y = 0$. Show further that

$$\text{either} \quad p = -2x \quad \text{or} \quad p = -(x^2+1)/x \quad \text{with} \quad x \neq 0\,.$$

Hence show that the set of points in the Argand diagram that can (as p varies) represent roots of the quadratic equation consists of the real axis with one point missing and a circle. This set of points is called the *root locus* of the quadratic equation.

(ii) Obtain and sketch in the Argand diagram the root locus of the equation

$$pz^2 + z + 1 = 0\,.$$

(iii) Obtain and sketch in the Argand diagram the root locus of the equation

$$pz^2 + p^2 z + 2 = 0\,.$$

STEP Mechanics question

5 Preparation

The basis of this question is just the standard calculation you do to find the range of a projectile on a plane so, to prepare for it, you should go over that calculation, as follows.

(i) A (small) cannon ball is fired from a cannon situated on a horizontal plane. Initially, its speed is u and its trajectory makes angle ϕ to the horizontal. Ignoring air resistance, show that the ball lands at time $2ug^{-1}\sin\phi$ after it is fired.

(ii) Show that it lands a distance $2u^2g^{-1}\sin\phi\,\cos\phi$ from the cannon.

(iii) Use calculus to show that this distance is greatest (for fixed u) when $\cos 2\phi = 0$.

(iv) Hence show that the range (i.e. the distance to the furthest point on the plane that it can reach) of the cannon is u^2g^{-1}.

For the STEP question, you follow these steps closely. The equations for the trajectory are the same, but the first variation comes almost immediately: the ball lands when $y = x\tan\beta$, not when $y = 0$. At each stage, you should check that your result agrees with the corresponding result above when $\beta = 0$.

6 The Mechanics question

A cannon is situated at the bottom of a plane inclined at angle β to the horizontal. A (small) cannon ball is fired from the cannon at an initial speed u. Ignoring air resistance, find the angle of firing which will maximise the distance up the plane travelled by the cannon ball and show that in this case the ball will land at a distance

$$\frac{u^2}{g(1+\sin\beta)}$$

from the cannon.

Discussion

You see from this question how quickly a mechanics question can descend (or ascend, according to your tastes) into mere algebra or coordinate geometry. Projectile questions are all about fitting a parabola to the given conditions.

I needed some trigonometrical formulae to get this one out. For example,

$$\cos\theta\cos A - \sin\theta\sin A = \cos(\theta + A) \qquad (1)$$

$$\sin(\tfrac{1}{4}\pi + \tfrac{1}{2}\beta) = \cos(\tfrac{1}{4}\pi - \tfrac{1}{2}\beta) \qquad (2)$$

$$2\sin^2 B = 1 - \cos(2B). \qquad (3)$$

You may not need them but, if you do, make sure that you know where they come from.

Finally, note that the distance referred to at the end of the question does not mean horizontal distance but distance up the plane. So when you have found the greatest value of x, you still have a little work to do.

STEP Probability/Statistics question

7 Preparation

You are probably familiar with discrete random variables, and may have met the Normal Distribution which is a special continuous random variable.

A continuous random variable has a *probability density function* which can be used to find the probability that the random variable's value falls in a particular range. The probability that the random function X lies in the range $a \le X \le b$ is given by:

$$\int_a^b \mathrm{f}(x)\mathrm{d}x$$

where $\mathrm{f}(x)$ is the probability density function.

Since the total probability has to be 1, the total area under the graph $y = \mathrm{f}(x)$ must be 1. You will sometimes have to use this fact, for example to find the value of a constant in $\mathrm{f}(x)$.

The *expectation* of a continuous random variable is given by:

$$\mathrm{E}(X) = \int_{-\infty}^{\infty} x \times \mathrm{f}(x)\mathrm{d}x\,.$$

The *Median* of a continuous random variable is the value M such that:

$$\int_{-\infty}^{M} \mathrm{f}(x)\mathrm{d}x = \int_{M}^{\infty} \mathrm{f}(x)\mathrm{d}x = \frac{1}{2}\,,$$

i.e. M splits the graph up so that the area each side is $\frac{1}{2}$. Remember that $\mathrm{f}(x)$ might be equal to zero for some ranges of x.

(i) A continuous random variable X has probability density function $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} kx & \text{for } 0 \leqslant x \leqslant 2 \\ 0 & \text{otherwise.} \end{cases}$$

(a) By using $\displaystyle\int_0^2 \mathrm{f}(x)\mathrm{d}x = 1$, find k.

(b) Find $\mathrm{P}(1 \leqslant X \leqslant 2)$.

(c) Find $\mathrm{E}(X)$.
You only have to integrate from $x = 0$ to $x = 2$ as the probability density function is zero elsewhere.

(d) Find the median, M, of X.

(ii) A continuous random variable X has probability density function $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} x^2 & \text{for } 0 \leqslant x \leqslant 1 \\ 1 & \text{for } 1 \leqslant x \leqslant \frac{5}{3} \\ 0 & \text{otherwise.} \end{cases}$$

(a) Verify that $\int_0^{\frac{5}{3}} \mathrm{f}(x)\,\mathrm{d}x = 1$.

You will need to do two separate integrals, one for $\frac{1}{2} \leqslant x \leqslant 1$ and one for $1 \leqslant x \leqslant \frac{4}{3}$.

(b) Find $\mathrm{P}\left(\frac{1}{2} \leqslant X \leqslant \frac{4}{3}\right)$.
Again you will need two integrals.

(c) Find $\mathrm{E}(X)$.
Again you will need two integrals.

(d) Find the median, M.

It would probably be a good idea to find $\displaystyle\int_0^1 \mathrm{f}(x)\mathrm{d}x$ first to see which side of $x = 1$ the median lies.

8 The Probability question

The continuous random variable X has probability density function $\mathrm{f}(x)$, where

$$\mathrm{f}(x) = \begin{cases} a & \text{for } 0 \leqslant x < k \\ b & \text{for } k \leqslant x \leqslant 1 \\ 0 & \text{otherwise,} \end{cases}$$

where $a > b > 0$ and $0 < k < 1$. Show that $a > 1$ and $b < 1$.

(i) Show that

$$\mathrm{E}(X) = \frac{1 - 2b + ab}{2(a - b)}\,.$$

(ii) Show that the median, M, of X is given by $M = \dfrac{1}{2a}$ if $a + b \geqslant 2ab$ and obtain an expression for the median if $a + b \leqslant 2ab$.

(iii) Show that $M < \mathrm{E}(X)$.

Discussion

For the very first part a well drawn sketch, with a few words of explanation, should be enough to show the required result. You will need to find k in terms of a and b.

For part **(ii)** the answers depend on which side of $x = k$ the median lies.

When trying to show that $M < \mathrm{E}(X)$, it will probably be easier to show that $M - \mathrm{E}(X) < 0$.