

STEP CORRESPONDENCE PROJECT

Assignment 27

STEP I question

1 Preparation

- (i) Use the substitution $y = xv$, where v is a function of x , to show that the differential equation:

$$x \frac{dy}{dx} - y = x^3 \quad (x \neq 0) \quad (*)$$

becomes:

$$\frac{dv}{dx} = x.$$

You get two terms when you use the product rule to differentiate xv .

Find v in terms of x and hence find y in terms of x . Substitute your solution for y into (*) to show that it does indeed satisfy the original equation.

- (ii) **Separation of Variables**

A differential equation of the form (or one that can be put into the form):

$$f(y) \frac{dy}{dx} = g(x) \quad (\dagger)$$

can be solved this by ‘separating the variables’. What you do is to write it as

$$f(y)dy = g(x)dx$$

and integrate both sides.

Although this is what you do in practice, it doesn’t really make much sense: what does the above equation mean??

To justify the method, we have to ask first what (\dagger) means. The solution to the equation will be a formula for y as a function of x , say $y = h(x)$. Therefore, we should really write (\dagger) as

$$f(h(x))h'(x) = g(x).$$

Integrating with respect to x gives

$$\int f(h(x))h'(x) dx = \int g(x) dx.$$

In Assignment 25 (Integration by Substitution) we saw that the left hand side of this equation is just what you get if you make the substitution $y = h(x)$ in the integral $\int f(y)dy$. We can therefore write the equation as

$$\int f(y)dy = \int g(x)dx,$$

in agreement with what we said in the first place.

Use this technique to solve:

(a) $x \frac{dy}{dx} = y + 1$, giving your answer in the form $y = \dots$.

Note that $e^{\ln x + c} = e^c \times e^{\ln x} = Ax$ where A is a constant such that $A = e^c$.

(b) $\frac{dy}{dx} = \tan y$.

Recall that $\tan y = \frac{\sin y}{\cos y}$.

(iii) Find A and B such that $\frac{x}{(x+1)(x+2)} = \frac{A}{x+1} + \frac{B}{x+2}$.

Hence evaluate $\int_1^2 \frac{x}{(x+1)(x+2)} dx$, giving your answer in the form $\ln\left(\frac{m}{n}\right)$.

2 The STEP I question

(i) Show that substituting $y = xv$, where v is a function of x , in the differential equation

$$xy \frac{dy}{dx} + y^2 - 2x^2 = 0 \quad (x \neq 0)$$

leads to the differential equation

$$xv \frac{dv}{dx} + 2v^2 - 2 = 0.$$

Hence show that the general solution can be written in the form

$$x^2(y^2 - x^2) = C,$$

where C is a constant.

(ii) Find the general solution of the differential equation

$$y \frac{dy}{dx} + 6x + 5y = 0 \quad (x \neq 0).$$

Discussion

Note that the first differential equations in y cannot be solved by separating the variables, so the point of making the substitutions is to achieve an equation for which we can separate the variables.

Part (ii) is very similar to part (i). As in part (i), you should leave your answer to part (ii) as an *implicit* equation; don't try to re-write it as $y = \dots$. However, you should write it in a form which does not involve fractions (make it look nice!).

STEP III question

3 Preparation

(i) Use the chain rule to differentiate $\sin(\ln x)$. Use this to help you differentiate $e^{2\sin(\ln x)}$.

(ii) Proof by Induction

The following shows an example of how to prove a conjecture by induction.

Question: Prove that $11^n - 4^n$ is divisible by 7 for integers $n \geq 1$.

When $n = 1$ (the ‘base case’) we have $11^1 - 4^1 = 7$, so the result is true when $n = 1$.

Assume the conjecture is true when $n = k$, so we have $11^k - 4^k = 7M$ for some integer M .

Now consider the case $n = k + 1$. We are required to prove (RTP) $11^{k+1} - 4^{k+1} = 7P$ for some integer P . You should expect to use the $n = k$ case at some point.

$$\begin{aligned} 11^{k+1} - 4^{k+1} &= 11 \times 11^k - 4 \times 4^k \\ &= 11 \times (7M + 4^k) - 4 \times 4^k \text{ using the } n = k \text{ case} \\ &= 11 \times 7M + 11 \times 4^k - 4 \times 4^k \\ &= 11 \times 7M + 7 \times 4^k \\ &= 7 \times (11M + 4^k) \\ &= 7P \text{ where } P \text{ is an integer.} \end{aligned}$$

Therefore, if it is true for $n = k$ it is true for $n = k + 1$.

But we know it is true for $n = 1$, and by taking $k = 1$, we find that it is true for $n = 2$. Now, taking $k = 2$, we find that it is true for $n = 3$. We can go on like this indefinitely¹ to show that it is true for any integer value of n .

Use induction to prove the following:

(a) $4^n + 6n - 1$ is divisible by 9 for any positive integer n .

(b) $\sum_{i=1}^n i^3 = \frac{1}{4}n^2(n+1)^2$ where $n \geq 1$.

It is very helpful when considering the $n = k + 1$ case to write down what you are required to prove (‘RTP’) so that you know what you are aiming for. For part (b) factorisation is your friend — don’t expand first and then have to factorise a quartic!

¹For the MEI A-levels you are supposed to write (for 2 marks): ‘Therefore if it is true for $n = k$ it is also true for $n = k + 1$. Since it is true for $n = 1$ it is true for all positive integers.’ This is a good thing to write: it shows that you understand the method (or that you can learn by heart a couple of sentences).

(iii) Maclaurin's expansion is:

$$f(x) = f(0) + \frac{x}{1!}f'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots,$$

(assuming this series converges).

Use this to find the first three non-zero terms in the expansion of $\sin x$. Hence estimate $\sin 0.1$ to seven decimal places, where 0.1 is in radians.

If 0.1 is in radians (not degrees), then we can use the result that the derivative of $\sin x$ is $\cos x$. You may well want to do the final calculation without a calculator, just because you can.

4 The STEP III question

Given that $y = \ln(x + \sqrt{x^2 + 1})$, show that $\frac{dy}{dx} = \frac{1}{\sqrt{x^2 + 1}}$.

Prove by induction that, for $n \geq 0$,

$$(x^2 + 1)y^{(n+2)} + (2n + 1)xy^{(n+1)} + n^2y^{(n)} = 0,$$

where $y^{(n)} = \frac{d^n y}{dx^n}$ and $y^{(0)} = y$.

Using this result in the case $x = 0$, or otherwise, show that the Maclaurin series for y begins

$$x - \frac{x^3}{6} + \frac{3x^5}{40}$$

and find the next non-zero term.

Discussion Note that the induction requires you to prove the result for $n \geq 0$, so make your 'base case' $n = 0$. Don't overlook the request tucked away at the end of the question (to find the next non-zero term)!

STEP Mechanics question

5 Preparation

- (i) If $\tan \theta = \frac{5}{12}$ and $0 \leq \theta < \frac{\pi}{2}$, find $\cos \theta$ and $\sin \theta$.
- (ii) A block of mass 20kg is lying on a slope which makes an angle of 30° to the horizontal. The coefficient of friction between the block and the slope is 0.7. Take $g = 10$. Find the minimum value needed to:
- (a) pull the block up the slope;
- (b) push the block down the slope.

Leave your answers in surd form.

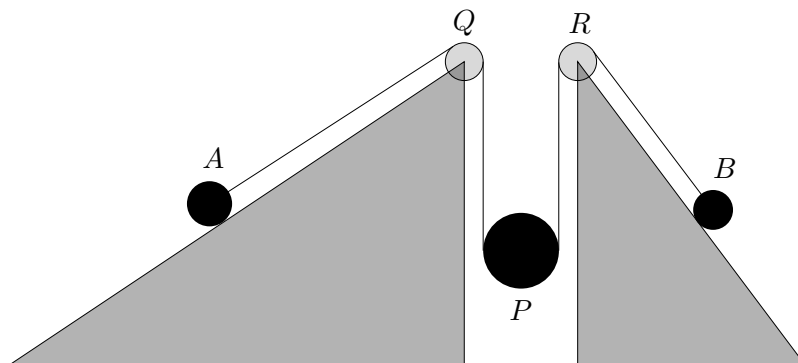
Draw a good big diagram, putting in all the forces! Remember that friction counteracts the tendency to move.

- (iii) Two weights, with masses m_1 and m_2 where $m_1 > m_2$, are connected by a light inextensible string over a smooth pulley. Find the acceleration of the weights in terms of m_1 , m_2 and g .

6 The Mechanics question

The diagram shows two particles, A of mass $5m$ and B of mass $3m$, connected by a light inextensible string which passes over two smooth, light, fixed pulleys, Q and R , and under a smooth pulley P which has mass M and is free to move vertically.

Particles A and B lie on fixed rough planes inclined to the horizontal at angles of $\arctan \frac{7}{24}$ and $\arctan \frac{4}{3}$ respectively. The segments AQ and RB of the string are parallel to their respective planes, and segments QP and PR are vertical. The coefficient of friction between each particle and its plane is μ .



- (i) Given that the system is in equilibrium, with both A and B on the point of moving up their planes, determine the value of μ and show that $M = 6m$.
- (ii) In the case when $M = 9m$, determine the initial accelerations of A , B and P in terms of g .

Discussion

Note that the second part of the question the three particles can have different accelerations (unlike question 5(iii)) but they are related because the length of the string is fixed.

STEP Probability/Statistics question**7 Preparation**

- (i) The expectation of a discrete random variable is defined by:

$$E(X) = \sum n \times P(X = n)$$

where the sum is taken over all possible values of n .

The discrete random variable X is defined by:

$$P(X = x) = \begin{cases} kx & \text{for } x = 1, 2, 3, 4, \\ 0 & \text{otherwise.} \end{cases}$$

- (a) Find the value of k .
- (b) Find $P(X \geq 3)$.
- (c) Find $E(X)$.
- (ii) I roll a normal six-sided die² until I roll a 6; then I stop.
- (a) Find the probability that I roll a 6 on the third roll.
- (b) Find that probability that I need more than five rolls to get a 6.

You could do $1 - (P(X = 1) + P(X = 2) + \dots + P(X = 5))$ but it is easier to think about what must have happened on the first five rolls if you need more than five rolls to get a 6.

- (iii) Find the value of the infinite sum $2 + 0.2 + 0.02 + \dots$.
- (iv) Find the maximum value of $y = x(4 - x)$.

²Apparently, the use of 'die' as the singular form of 'dice' is normal in America and in maths books, and certainly on STEP papers. But, at least in conversation, the British never say die.

8 The Probability question

A discrete random variable X takes only positive integer values. Define $E(X)$ for this case, and show that

$$E(X) = \sum_{n=1}^{\infty} P(X \geq n).$$

I am collecting toy penguins from cereal boxes. Each box contains either one daddy penguin or one mummy penguin. The probability that a given box contains a daddy penguin is p and the probability that a given box contains a mummy penguin is q , where $p \neq 0$, $q \neq 0$ and $p + q = 1$.

Let X be the number of boxes that I need to open to get at least one of each kind of penguin. Show that $P(X \geq 4) = p^3 + q^3$, and that

$$E(X) = \frac{1}{pq} - 1.$$

Hence show that $E(X) \geq 3$.

Discussion Even if you cannot do the first part of the question, you can still do the rest. It may be helpful to note that, for example, $3 \times P(X = 3) = P(X = 3) + P(X = 3) + P(X = 3)$.

For $P(X \geq 4)$ you could consider $1 - P(X \leq 3)$ but this is probably not the easiest way.

For $E(X)$ be careful with the $n = 1$ case, it does not fit the same pattern as the other values of n .