STEP CORRESPONDENCE PROJECT

Assignment 28

STEP I question

1 Preparation

(i) Find an expression for the infinite sum $1 + p + p^2 + \cdots$, where $-1 < p < 1$. Use this to show that $1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots = 2$.

(ii) By differentiating your expression in part **(i)** with respect to p, find an expression for:

$$1 + 2p + 3p^2 + \cdots .$$

Hence evaluate $1 + \frac{2}{2} + \frac{3}{4} + \frac{4}{8} + \frac{5}{16} + \cdots$.

(iii) Expand $(1 + x)^5$. By integrating this result, show that:

$$x + \tfrac{5}{2}x^2 + \tfrac{10}{3}x^3 + \tfrac{5}{2}x^4 + x^5 + \tfrac{1}{6}x^6 + c \equiv \tfrac{1}{6}(1 + x)^6 ,$$

where c is a constant of integration. By choosing a value for x, find c.

2 The STEP I question

By considering the expansion of $(1 + x)^n$ where n is a positive integer, or otherwise, show that:

(i) $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$;

(ii) $\binom{n}{1} + 2\binom{n}{2} + 3\binom{n}{3} + \cdots + n\binom{n}{n} = n2^{n-1}$;

(iii) $\binom{n}{0} + \frac{1}{2}\binom{n}{1} + \frac{1}{3}\binom{n}{2} + \cdots + \frac{1}{n+1}\binom{n}{n} = \frac{1}{n+1}\left(2^{n+1} - 1\right)$;

(iv) $\binom{n}{1} + 2^2\binom{n}{2} + 3^2\binom{n}{3} + \cdots + n^2\binom{n}{n} = n\,(n+1)\,2^{n-2}$.

Note: $\binom{n}{r}$ is a notation for $\dfrac{n!}{r!(n-r)!}$ and the binomial expansion is

$$(1 + x)^n = \binom{n}{0} + \binom{n}{1}x + \binom{n}{2}x^2 + \cdots + \binom{n}{n}x^n .$$

STEP III question

3 Preparation

(i) Although you probably won't study limits rigorously until you get to university, there are various properties of limits that you can understand and use without formal definitions.

For example, if $\lim_{x\to a} \mathrm{f}(x) = L$ and $\lim_{x\to a} \mathrm{g}(x) = K$ (where L and K are **finite**) then:

i. $\lim_{x\to a} c\mathrm{f}(x) = c \lim_{x\to a} \mathrm{f}(x) = cL$

ii. $\lim_{x\to a} \big(\mathrm{f}(x) + \mathrm{g}(x)\big) = \lim_{x\to a} \mathrm{f}(x) + \lim_{x\to a} \mathrm{g}(x) = L + K$

iii. $\lim_{x\to a} \big(\mathrm{f}(x)\mathrm{g}(x)\big) = \lim_{x\to a} \mathrm{f}(x) \times \lim_{x\to a} \mathrm{g}(x) = L \times K$

iv. $\lim_{x\to a} \dfrac{\mathrm{f}(x)}{\mathrm{g}(x)} = \dfrac{\lim_{x\to a} \mathrm{f}(x)}{\lim_{x\to a} \mathrm{g}(x)} = \dfrac{L}{K} \quad (K \neq 0).$

v. If $\mathrm{h}(x)$ is continuous, then $\lim_{x\to a} \mathrm{h}\big(\mathrm{g}(x)\big) = \mathrm{h}\big(\lim_{x\to a} \mathrm{g}(x)\big) = \mathrm{h}(K)\,.$

These all make good sense, and you would probably have assumed them to be true.

Evaluate the following limits, stating which of the above properties you have used:

(a) $\lim_{\theta\to 0} \dfrac{\sin\theta}{\cos\theta}$

(b) $\lim_{x\to\infty} 5(\mathrm{e}^{-x} + 1)$

(c) $\lim_{t\to\infty} \cos\frac{1}{t}$

(d) $\lim_{x\to\infty} \mathrm{e}^{\frac{1}{x}}$

(e) $\lim_{x\to 0} \mathrm{e}^{-\frac{1}{x}} \quad (x > 0).$

(ii) By first writing $\mathrm{e}^{-t}t$ as $\dfrac{t}{\mathrm{e}^t}$ and then using the series expansion for e^t show that:

$$\lim_{t\to\infty} \mathrm{e}^{-t}t = 0\,.$$

There are various ways you can do this. You could, for example, start by dividing top and bottom by t.

Show that $\lim_{t\to\infty} e^{-t}t^n = 0$ where n is a positive integer.

(iii) If $x = \mathrm{e}^{-t}$, how does x behave as $t \to \infty$? By setting substitution $x = \mathrm{e}^{-t}$ in the expression $x\ln x$, show that $\lim_{x\to 0}(x\ln x) = 0$.

(iv) Consider the integral $\displaystyle\int_0^1 \ln x \, \mathrm{d}x$. The function $\ln x$ is not defined when $x = 0$, but the integral can be defined as a limit, as follows.

(a) Evaluate $\displaystyle\int_r^1 \ln x \, \mathrm{d}x$ (see Assignment 24 if you are stuck).

(b) Now take the limit of this result as $r \to 0$.

4 The STEP III question

Let m be a positive integer and let n be a non-negative integer. Use the result $\lim\limits_{t\to\infty} \mathrm{e}^{-mt}t^n = 0$ to show that

$$\lim_{x\to 0} x^m (\ln x)^n = 0 \,.$$

(i) By writing x^x as $\mathrm{e}^{x \ln x}$ show that

$$\lim_{x\to 0} x^x = 1 \,.$$

(ii) Let $I_n = \displaystyle\int_0^1 x^m (\ln x)^n \mathrm{d}x$. Show that

$$I_{n+1} = -\frac{n+1}{m+1} I_n$$

and hence evaluate I_n.

(iii) Show that

$$\int_0^1 x^x \mathrm{d}x = 1 - \left(\tfrac{1}{2}\right)^2 + \left(\tfrac{1}{3}\right)^3 - \left(\tfrac{1}{4}\right)^4 + \cdots \,.$$

Discussion

Who would have thought the final integral would come out as such an interesting-looking series?

For part **(ii)** start by considering I_{n+1} and then integrating this (using a standard method we have covered recently). For part **(iii)** start by writing x^x as in part **(i)**.

A small point: towards the end of part **(iii)** you have to integrate an infinite series, which you do by integrating each term and adding them all up. You have used the result that the integral of a sum is equal to the sum of the integrals, which certainly holds for the sum of a finite number of terms. However, it may not hold for an infinite number of terms; it depends on how well the series converges (as you will discover in your first university year). We are OK here, because the exponential series converges splendidly well, and you weren't intended to worry about it.

You might wonder why you are asked to find I_{n+1} in part (**(ii)** instead of what you actually want, which is I_n. The reason is probably so that the constraint on n given in the first line of the question works. Since $n = 0$ is allowed, the reduction formula $I_n = \cdots$ would have I_{-1} on the right hand side when $n = 0$, which is not defined (and anyway the method breaks down because for this integral).

STEP Mechanics question

5 Preparation

We need some basic ideas to tackle this question:

- If you express the position of a particle (of mass m) at time t as the position vector $\begin{pmatrix} x \\ y \end{pmatrix}$, then the velocity (a vector, of course) is obtained by differentiation: $\mathbf{v} = \begin{pmatrix} \dot{x} \\ \dot{y} \end{pmatrix}$, where the dot denotes differentiation with respect to t.
- If you know the velocity in the above form, then the speed of the particle is just the length of the velocity vector, which is $\sqrt{\dot{x}^2 + \dot{y}^2}$. The kinetic energy is then $\frac{1}{2}m(\dot{x}^2+\dot{y}^2)$. It looks as if you are adding the kinetic energy in the x direction to the kinetic energy in the y direction, but you shouldn't think of it like this: energy is not a vector, so doesn't have components in different directions.
- The potential energy of a particle of mass m in a uniform gravitational field (the gravitational field of the Earth can be considered to be uniform over relatively short distances) can be written as mgy where y is the height of the particle above some chosen level.
- Total energy for a system of particles is the sum of the kinetic and potential energy of each particle.
- If there are no dissipative forces (friction, for example), then the total energy of a system is conserved — which means that it is constant. If you differentiate this energy with respect to time, you get zero.

(i) A particle moves on the circle $x^2 + (y - a)^2 = a^2$. It starts at the origin. Draw a diagram and write down x and y coordinates of the particle, in terms of a and θ, when the radius from the centre to the particle makes an angle θ with the downward vertical (assume $x \leqslant 0$). Obtain the velocity vector of the particle given that $\dot{\theta} = \omega$. Show that the speed of the particle is $a\omega$ (for $\omega > 0$).

(ii) A thin circular hoop of radius a rolls without slipping in a straight line on a horizontal plane, with the plane of the hoop vertical. You are given that $\dot{\theta} = \omega$, which is constant. When the hoop has rolled through θ radians, how far has its centre moved? Find the position vector and the velocity vector of the centre.

6 The Mechanics question

Two identical particles P and Q, each of mass m, are attached to the ends of a diameter of a light thin circular hoop of radius a. The hoop rolls without slipping along a straight line on a horizontal table with the plane of the hoop vertical. Initially, P is in contact with the table. At time t, the hoop has rotated through an angle θ. Write down the position at time t of P, relative to its starting point, in cartesian coordinates, and determine its speed in terms of a, θ and $\dot{\theta}$. Show that the total kinetic energy of the two particles is $2ma^2\dot{\theta}^2$.

Given that the only external forces on the system are gravity and the vertical reaction of the table on the hoop, show that the hoop rolls with constant speed.

STEP Probability/Statistics question

7 **Preparation**

This question involves *arrangements*, and you may like to look back to Assignment 6 before you start. You can leave big numbers in terms of factorials.

(i) 6 women (A, B, C, D, E and F) stand in a line. In how many ways can they arrange themselves in this line?

There are 6 different positions for A, then 5 for B etc.

(ii) The 6 women are joined by 4 men (G, H, I and J). In how many ways can the 10 people arrange themselves?

(iii) The 4 men wish to stand together. We can imagine them being together inside a rope ring.

(a) In how many ways can the 4 men arrange themselves in a line inside their rope?

(b) Imagine now that the rope is one "Person" so that there are 7 "people" i.e. the 6 women and the 1 roped-off collection of men. How many ways can we arrange these 7 "people"?

(c) How many ways are there of arranging the 10 people so that the men stand together?

(d) Show that the probability that all 4 men stand together is 1/30.

(iv) What is the probability that all the women (but not necessarily the men) stand together?

(v) What is the probability that all the men stand together and all the women stand together?

Note that this is **not** the previous two answers multiplied together as the events are not independent.

(vi) Find the probability that no men stand together by following these steps.

(a) First arrange the 6 women. How many ways can we do this?

(b) There are now 7 gaps between the women which we can slot the first man into, like this:

$$\downarrow W_1 \downarrow W_2 \downarrow W_3 \downarrow W_4 \downarrow W_5 \downarrow W_6 \downarrow$$

where the arrows represent the gaps. Once a man fills up a gap then it is no longer a gap and there are only 6 gaps for the second man. Find the number of ways you can arrange the 4 men in the 7 gaps.

(c) Find hence find the probability that no two men are together.

(vii) Find the probability that there will be a woman at each end of the line, by following these steps.

(a) How many different ways can we choose the two women to be at the ends? (Consider how many choices you have for the first end and then how many for the second end).

(b) How many ways can you arrange the 8 people left?

(c) Use your two previous answers to find the probability that there is a woman at each end of the line.

8 The Probability question

I seat n boys and 3 girls in a line at random, so that each order of the $n+3$ children is as likely to occur as any other. Let K be the maximum number of consecutive girls in the line so, for example, $K = 1$ if there is at least one boy between each pair of girls.

(i) Find $\mathrm{P}(K = 3)$.

(ii) Show that

$$\mathrm{P}(K = 1) = \frac{n(n-1)}{(n+2)(n+3)} .$$

(iii) Find $\mathrm{E}(K)$.

Discussion

You might like to simplify your final answer, though the question doesn't ask you to do so. More overleaf ...

If you found this question quite straightforward (after the preparation!), you might like to try this one which you can hand in (as well as your other two questions) if you like:

9 A school has n pupils, of whom r play hockey, where $n \geqslant r \geqslant 2$. All n pupils are arranged in a row at random.

(i) What is the probability that there is a hockey player at each end of the row?

(ii) What is the probability that all the hockey players are standing together?

(iii) By considering the gaps between the non-hockey-players, find the probability that no two hockey players are standing together, distinguishing between cases when the probability is zero and when it is non-zero.