STEP CORRESPONDENCE PROJECT

Assignment 29

STEP I question

1 Preparation

Any even number can be written in the form $2k$ and any odd number can be written in the form $2k+1$, where k is an integer.

(i) By considering $(2a+1)(2b+1)$, prove that the product of any two odd numbers is an odd number.

Note that $2a+1$ and $2b+1$ represent any two odd numbers, even the same odd numbers if $a=b$. A common mistake, when tackling this sort of question, is to use $(2n+1)$ and $(2n-1)$ but then you have only shown that the product of two *consecutive* odd numbers is odd.

(ii) Prove that the sum of two odd numbers is an even number.

(iii) Simplify $(2k+1)^2 - (2k-1)^2$ and hence prove that any number of the form $8k$ can be written as the difference of two odd squares.

(iv) Show that $(2a+1)^2 - (2b+1)^2 = 4(a-b)(a+b+1)$. Hence prove that the difference between any two odd squares is divisible by 8.

Exactly one of the two brackets $(a-b)$ and $(a+b+1)$ is even. You should explain why this is true. Note that the direction of proof here is opposite to that in part **(iii)**.

(v) State and prove a conjecture about the difference of two even squares.

(vi) Write down the value of $58^2 - 42^2$.

(vii) If $ab = 12$, $a > b$, and a and b are non-negative integers, find the possible values of a and b.

2 The STEP I question

All numbers referred to in this question are non-negative integers.

(i) Express each of the numbers 3, 5, 8, 12 and 16 as the difference of two non-zero squares.

(ii) Prove that any odd number can be written as the difference of two squares.

(iii) Prove that all numbers of the form $4k$, where k is a non-negative integer, can be written as the difference of two squares.

(iv) Prove that no number of the form $4k + 2$, where k is a non-negative integer, can be written as the difference of two squares.

(v) Prove that any number of the form pq, where p and q are prime numbers greater than 2, can be written as the difference of two squares in exactly two distinct ways. Does this result hold if p is a prime greater than 2 and $q = 2$?

(vi) Determine the number of distinct ways in which 675 can be written as the difference of two squares.

Discussion

For part **(iv)**, if $a^2 - b^2$ is your difference of two squares, you can consider the 4 cases depending on whether a and/or b is odd/even. You have already considered two of these cases in question **1**.

It is difficult to write a preparation for this question which does not give too much away. If you would like extra hints, please use the forums.

STEP III question

3 Preparation

The hyperbolic functions $\sinh x$ (pronounced "sinch or shine") and $\cosh x$ are defined by:

$$\sinh x = \frac{e^x - e^{-x}}{2} \quad \text{and} \quad \cosh x = \frac{e^x + e^{-x}}{2}\,.$$

(i) Show that $\dfrac{d}{dx}(\cosh x) = \sinh x$ and find $\dfrac{d}{dx}(\sinh x)$.

(ii) Sketch $\cosh x$ and $\sinh x$.

Note that when $x \to +\infty$, we can ignore the e^{-x} term.

(iii) Simplify $\cosh x + \sinh x$ and $\cosh x - \sinh x$.

(iv) Simplify $(\cosh x)^2 - (\sinh x)^2$.

(v) Solve the equation $\dfrac{dy}{dx} = \tanh y$, where $\tanh y = \dfrac{\sinh y}{\cosh y}$ and is pronounced "tanch" or "th-an". Write your answer in the form $\sinh y = \cdots$.

4 The STEP III question

(i) Solve the equation $u^2 + 2u \sinh x - 1 = 0$ giving u in terms of x.

Find the solution of the differential equation

$$\left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx}\sinh x - 1 = 0$$

that satisfies $y = 0$ and $\dfrac{dy}{dx} > 0$ at $x = 0$.

(ii) Find the solution, not identically zero, of the differential equation

$$\sinh y \left(\frac{dy}{dx}\right)^2 + 2\frac{dy}{dx} - \sinh y = 0$$

that satisfies $y = 0$ at $x = 0$, expressing your solution in the form $\cosh y = f(x)$. Show that the asymptotes to the solution curve are $y = \pm(-x + \ln 4)$.

Discussion

Asymptotes are lines that the curve approaches as either x or y or both tend to infinity. As some terms get very big, other terms (such as constant terms) can be ignored. For example, as $x \to +\infty$, $\sqrt{x^2 + 1} \approx \sqrt{x^2} = x$.

STEP Mechanics question

5 Preparation

Generally, and rather surprisingly, momentum is normally conserved in a collision (even in an explosion). But considering the conservation of momentum is not usually enough to determine the motion after the collision. For example, in one dimension following a collision of two particles, there are two unknowns, namely the velocities of the two particles. Conservation of momentum gives just one equation, so we need one more equation. This is supplied by what is known as Newton's experimental law.

Newton's Experimental law is:

$$e = \frac{v_B - v_A}{u_A - u_B}$$

where u_A and u_B are the initial velocities of the two particles and v_A and v_B are the final velocities.[1] Think of this as

$$e = \frac{\text{Speed of Separation}}{\text{Speed of Approach}}.$$

Note that these velocities can both be positive or both negative (if the particles are moving in the same direction) or they can have different signs if they are moving towards each other.[2] Here e is a constant which depends on the particles, called the *coefficient of restitution*. It describes how the speed of separation of two particles after a collision is related to their speed of approach.

Normally, kinetic energy is lost is a collision, being converted to heat or sound. A perfectly *elastic* collision is one where $e = 1$, and none of the kinetic energy is dissipated. A perfectly *inelastic* collision has $e = 0$ and the particles stick together. All collisions have $0 \leqslant e \leqslant 1$.

(i) Two particles have a head on collision (which means they are travelling in opposite directions). If the first particle has mass 5kg and initial speed 3ms^{-1} and the second particle has mass 2kg and initial speed 5ms^{-1} and the coefficient of restitution is $e = 0.5$ find an equation connecting v_A and v_B.

Be careful: the particles are initially travelling in opposite directions. You need to decide which direction is going to be positive. A good diagram of before and after will be helpful.

(ii) Given that linear momentum is conserved, write down another equation connecting v_A and v_B. Hence find the final speeds.

(iii) Show that the change in kinetic energy resulting from the collision is $\dfrac{1680}{49}$ Joules. (You may find the difference of squares (twice) useful. Or you may not.)

---

[1] We call them velocities, even though we are only considering motion in one dimension (along a line), because they can be going in either direction (to the left or to the right). It is normal to use the term 'speed' as the magnitude of velocity, so it is never negative.

[2] The signs can get confusing; it is best to rely on common sense rather than formulae.

6 The Mechanics question

Three particles P_1, P_2 and P_3 of masses m_1, m_2 and m_3 respectively lie at rest in a straight line on a smooth horizontal table. P_1 is projected with speed v towards P_2 and brought to rest by the collision. After P_2 collides with P_3, the latter moves forward with speed v. The coefficients of restitution in the first and second collisions are e and e', respectively. Show that

$$e' = \frac{m_2 + m_3 - m_1}{m_1}.$$

Show that $2m_1 \geqslant m_2 + m_3 \geqslant m_1$ for such collisions to be possible.

If m_1, m_3 and v are fixed, find, in terms of m_1, m_3 and v, the largest and smallest possible values for the final energy of the system.

It is important to think of a good way of naming the velocities of the particles before and after each collision; this of course must be carefully set out in your answer.

Don't forget, when you are trying to derive the inequalities, that $0 \leqslant e \leqslant 1$.

Questions 2009 S1 Q11 and 2011 S2 Q9 are similar (but don't send them in!).

STEP Probability/Statistics question

7 **Preparation**

(i) By first writing $9^x = (3^2)^x$, solve $3^{2x} - 3^x - 2 = 0$, giving your answer(s) in terms of logarithms.

(ii) Solve the inequality $1 - \sqrt{1-x} > x$.

Squaring both sides can be a valid method but *only* in certain circumstances such as when you know both sides are positive.

The *Poisson* distribution measures the number of occurrences of an event in a given time interval. It was first used by Ladislaus Josephovich Bortkiewicz to model the number of deaths of Prussian cavalry-men by horse kicks in a year.

A Poisson random variable satisfies the following conditions:

I Occurrences are independent.

II The probability of an occurrence during any time interval is proportional to the duration of the time interval.

As well as modelling the number of occurrences in a given time interval it can be used to model the number of occurrences in a given space interval. Some applications are the number of car accidents in a mile of road, the number of people joining a queue every 5 minutes and the number of hairs in a burger.

The number of occurrences in a given time interval is given by:

$$\mathrm{P}(X = n) = \frac{\mathrm{e}^{-\lambda}\lambda^n}{n!}$$

where n is an integer, with $n \geqslant 0$, λ is the mean number of occurrences in the given interval and (by convention) $0! = 1$.

Note that the sum of all the probabilities is given by:

$$\sum_{n=0}^{\infty} \frac{\mathrm{e}^{-\lambda}\lambda^n}{n!} = \mathrm{e}^{-\lambda} \times \sum_{n=0}^{\infty} \frac{\lambda^n}{n!} = \mathrm{e}^{-\lambda} \times \mathrm{e}^{\lambda} = 1\,.$$

For the last equality, we used the exponential series. You may like to show that $\mathrm{E}(X) = \lambda$.

(iii) The number of patients arriving at a drop-in chocoholics centre follows a Poisson distribution with an average rate of 5 an hour (so $\lambda = 5$). Don't bother to evaluate the exponentials or powers or factorials in the following questions.

(a) Find the probability that exactly 3 patients arrive between 9am and 10am.

(b) Find the probability that at least 1 patient arrives between 10am and 11am.

(c) The centre closes for lunch between 1pm and 2pm. It then runs for 3 more hours in the afternoon. Find the probability that no-one turns up for the first hour after lunch, then exactly 2 people turn up between 3pm and 4pm and then exactly 6 people turn up in the hour before closing.

Disjoint time intervals are independent, so you can multiply the probabilities.

(iv) If the number of occurrences in a time interval can be modelled as a Poisson distribution with a mean of λ, then the number of occurrences in a time interval twice as long can be modelled as a Poisson distribution with mean 2λ, and so on.

The number of calls to "Kalculus Kids" is Poisson distributed with, on average 8 calls an hour. Find the probabilities of:

(a) exactly 5 calls in one hour.

(b) exactly 10 calls in two hours.

(c) exactly 2 calls in 30 mins.

(d) Fewer than 2 calls in 15 mins.

(v) If you have two *independent* Poisson random variables, X with mean λ and Y with mean μ, then the random variable $X + Y$ also has a Poisson distribution, with mean $\lambda + \mu$.

A company making gizmos accepts orders on-line or by phone. Both of these follow a Poisson distribution: telephone orders with a mean of 2 per day; and on-line orders with a mean of 5 per day.

(a) What is the probability that they receive 10 orders on one day?

(b) What is the probability that they receive fewer than 3 orders in one day?

8 The Probability question

The number of texts that George receives on his mobile phone can be modelled by a Poisson random variable with mean λ texts per hour. Given that the probability George waits between 1 and 2 hours in the morning before he receives his first text is p, show that

$$pe^{2\lambda} - e^{\lambda} + 1 = 0.$$

Given that $4p < 1$, show that there are two positive values of λ that satisfy this equation.

The number of texts that Mildred receives on each of her two mobile phones can be modelled by independent Poisson random variables with different means λ_1 and λ_2 texts per hour. Given that, for each phone, the probability that Mildred waits between 1 and 2 hours in the morning before she receives her first text is also p, find an expression for $\lambda_1 + \lambda_2$ in terms of p.

Find the probability, in terms of p, that she waits between 1 and 2 hours in the morning to receive her first text.

Discussion

Note that $\lambda > 0$ if and only if $e^{\lambda} > 1$.