STEP CORRESPONDENCE PROJECT

Assignment 30

STEP I question

1 Preparation

(i) If $\tan 3\theta = 1$, what are the possible values of θ in the range $0 \leqslant \theta < 2\pi$?

(ii) If $\sin\theta = \frac{1}{2}$ and $\cos\theta < 0$, what are the possible values of θ?

(iii) Show that $\sin 3\theta = 3\sin\theta\cos^2\theta - \sin^3\theta$. By replacing θ by $\frac{1}{2}\pi - \theta$, deduce from this result an expression for $\cos 3\theta$.

(iv) By spotting one solution (which may not be an integer), and using it to reduce the cubic to a quadratic, solve the equations:

(a) $x^3 - 2x^2 - 11x + 12 = 0$
(b) $12x^3 + 11x^2 - 2x - 1 = 0$
(c) $2x^3 - 5x^2 + 1 = 0$

(v) Show that $-1 < 8 - 5\sqrt{3} < 0$.

2 The STEP I question

(i) Given that $\cos\theta = \dfrac{3}{5}$ and that $\dfrac{3\pi}{2} \leqslant \theta \leqslant 2\pi$, show that $\sin 2\theta = -\dfrac{24}{25}$, and evaluate $\cos 3\theta$.

(ii) Prove the identity $\tan 3\theta \equiv \dfrac{3\tan\theta - \tan^3\theta}{1 - 3\tan^2\theta}$.

Hence evaluate $\tan\theta$, given that $\tan 3\theta = \dfrac{11}{2}$ and that $\dfrac{\pi}{4} \leqslant \theta \leqslant \dfrac{\pi}{2}$.

STEP III question

3 Preparation

(i) **(a)** Prove by induction that $(\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$.

(b) Find all solutions of the equation $z^5 = 1$, by writing z as $r(\cos\theta + i\sin\theta)$, where $r \geqslant 0$. You should justify each step carefully.

(c) Express $\cos 5\theta$ in terms of $\cos\theta$ and $\sin\theta$.

(ii) If the roots of the equation $x^n + a_1x^{n-1} + \cdots + a_{n-1}x + a_n = 0$, are α_1, α_2, $\ldots$, α_n, what is the expression for the product of the roots in terms of the coefficients a_i?

It just involves one coefficient, and depends on n. If you haven't seen this before, factorise the equation in the form $(x - \alpha_1)\cdots(x - \alpha_n) = 0$

4 The STEP III question

By considering the imaginary part of the equation $z^7 = 1$, or otherwise, find all the roots of the equation

$$t^6 - 21t^4 + 35t^2 - 7 = 0.$$

You should justify each step carefully.

Hence, or otherwise, prove that

$$\tan\frac{2\pi}{7}\tan\frac{4\pi}{7}\tan\frac{6\pi}{7} = \sqrt{7}.$$

Find the corresponding result for

$$\tan\frac{2\pi}{n}\tan\frac{4\pi}{n}\cdots\tan\frac{(n-1)\pi}{n}$$

in the two cases $n = 9$ and $n = 11$.

STEP Mechanics question

5 Preparation

A particle of mass m moves on a smooth horizontal surface. It experiences a constant force of magnitude F in the positive x-direction. Initially, the particle is at the origin and is moving with speed u in a direction that makes an angle of α with the positive x-axis ($\alpha = \frac{1}{2}\pi$ corresponds to the positive y-axis).

(i) Write down expressions for $\ddot{x}$ and $\ddot{y}$ and, by integration and using the initial conditions, obtain the position of the particle at time t.

(a) If $\alpha = -\pi$, find the time taken for the particle to return to the origin.
(b) If $\alpha = -\frac{3}{4}\pi$, find the time taken for the particle to reach the y-axis.
(c) If $\alpha = \frac{1}{2}\pi$, find the time taken for the particle to reach the line $y = x$.

(ii) Write $g \sin 2\theta + f \cos 2\theta$ in the form $R \sin(2\theta + \beta)$, and hence find its maximum value.

6 The Mechanics question

A projectile of unit mass is fired in a northerly direction from a point on a horizontal plain at speed u and an angle θ above the horizontal. It lands at a point A on the plain. In flight, the projectile experiences two forces: gravity, of magnitude g; and a horizontal force of constant magnitude f due to a wind blowing from North to South. Derive an expression, in terms of u, g, f and θ for the distance OA.

(i) Determine the angle α such that, for all $\theta > \alpha$, the wind starts to blow the projectile back towards O before it lands at A.

'Blow the projectile back towards O' means that the projectile's horizontal velocity is towards the origin.

(ii) An identical projectile, which experiences the same forces, is fired from O in a northerly direction at speed u and angle $45°$ above the horizontal and lands at a point B on the plain. Given that θ is chosen to maximise OA, show that

$$\frac{OB}{OA} = \frac{g - f}{\sqrt{g^2 + f^2} - f} \,.$$

Describe carefully the motion of the second projectile when $f = g$.

STEP Probability/Statistics question

7 **Preparation**

(i) A group of 5 children, aged 2, 4, 6, 8 and 10, line up randomly (so that each order in the line is equally likely) for birthday cake.

(a) Find the probability that the youngest child is first in the queue.

(b) Find the probability that the second child in the queue is the oldest given that the first child is the youngest.

(ii) I create a four-digit number by placing the digits $1, 2, 3$ and 4 in random order (each order being equally likely).

(a) How many such numbers can I create?

(b) How many of these numbers are divisible by 4?
There are various ways of doing this — the low-tech approach is to write out all the different options.

(c) What is the probability that I create a number bigger than 2413?

(d) If the first digit of my number is 1, what is the probability that the number is divisible by 4?

8 I know that ice-creams come in n different sizes, but I don't know what the sizes are. I am offered one of each in succession, in random order. I am certainly going to choose one - the bigger the better - but I am not allowed more than one. My strategy is to reject the first ice-cream I am offered and choose the first one thereafter that is bigger than the first one I was offered; if the first ice-cream offered is in fact the biggest one, then I have to put up with the last one, however small.

Let $\mathrm{P}_n(k)$ be the probability that I choose the kth biggest ice-cream, where $k = 1$ is the biggest and $k = n$ is the smallest.

(i) Show that $\mathrm{P}_4(1) = \frac{11}{24}$ and find $\mathrm{P}_4(2)$, $\mathrm{P}_4(3)$ and $\mathrm{P}_4(4)$.

(ii) Find an expression for $\mathrm{P}_n(1)$.

Discussion

The expression in part **(ii)** will be in the form of a sum. If you need a starting point then please use the forums.