STEP CORRESPONDENCE PROJECT

Assignment 31

STEP I question

1 Preparation

(i) Point A has position vector $\mathbf{a}$ (i.e. $\overrightarrow{OA} = \mathbf{a}$), and point B has position vector $\mathbf{b}$.

In the following questions, you will find diagrams useful.

(a) Find (in terms of $\mathbf{a}$ and $\mathbf{b}$) the vector $\overrightarrow{AB}$.

(b) Find the position vector of point C which lies $\frac{2}{3}$ of the way along the line segment AB (so that C is closer to B than to A).

(c) Draw a diagram showing the location of the point with position vector $\mathbf{a}+\lambda(\mathbf{b}-\mathbf{a})$ in the two cases $0 < \lambda < 1$ and $\lambda > 1$.

(ii) The binary[1] operation $\oplus$ is defined by:

$$a \oplus b = ab - a\,.$$

(a) Find $2 \oplus 3$ and $3 \oplus 2$.

(b) Find the conditions for which $a \oplus b$ is equal to $b \oplus a$.

(c) Find $3 \oplus (5 \oplus 1)$ and $(3 \oplus 5) \oplus 1$.

(d) Find the conditions under which $a \oplus (b \oplus c)$ and $(a \oplus b) \oplus c$ are distinct; i.e. $a \oplus (b \oplus c) \neq (a \oplus b) \oplus c$.

[1] A binary operation is one that acts on two objects. Multiplication, subtraction, vector dot product are all examples.

2 The STEP I question

For any two points X and Y, with position vectors $\mathbf{x}$ and $\mathbf{y}$ respectively, $X * Y$ is defined to be the point with position vector $\lambda\mathbf{x} + (1-\lambda)\mathbf{y}$, where λ is a fixed number.

(i) If X and Y are distinct, show that $X * Y$ and $Y * X$ are distinct unless λ takes a certain value (which you should state).

(ii) Under what conditions are $(X * Y) * Z$ and $X * (Y * Z)$ distinct?

(iii) Show that, for any points X, Y and Z,

$$(X * Y) * Z = (X * Z) * (Y * Z)$$

and obtain the corresponding result for $X * (Y * Z)$.

(iv) The points P_1, P_2, ... are defined by $P_1 = X * Y$ and, for $n \geqslant 2$, $P_n = P_{n-1} * Y$. Given that X and Y are distinct and that $0 < \lambda < 1$, find the ratio in which P_n divides the line segment XY.

Discussion

This question looks particularly frightening, because it involves not only vectors but a completely new operation defined in the question. However, those who were brave enough to try it in the examination were usually well rewarded.

The sticking point was often the last request in part (iii); here you have to take a shot in the dark (always tricky under examination conditions), but a calm look at the previous result should suggest something (which you would then have to verify).

Part (iv) could be done geometrically or (if you work out P_2 and P_3, say, and make a conjecture) by induction.

STEP III question

3 **Preparation**

In geometry the *centroid* of a set of points is, roughly speaking, the average position position of the points. If particles of equal mass were placed at each point, the centroid would coincide with their centre of mass (or centre of gravity). In vector language, if the points have position vectors $\mathbf{p}_1$, $\mathbf{p}_2$, ..., $\mathbf{p}_n$, then the centroid is at $(\mathbf{p}_1 + \mathbf{p}_2 + \cdots + \mathbf{p}_n)/n$.

The three vertices P_1, P_2 and P_3 of an equilateral triangle lie on a circle of radius 1, and their position vectors with respect to an origin at the centre of the circle are $\mathbf{p}_1$, $\mathbf{p}_2$ and $\mathbf{p}_3$.

(i) Explain without calculation, using a symmetry argument, why $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 = \mathbf{0}$.
This takes a bit of thought; or, rather, a bit of thought to come up with a careful explanation. Even if you are not sure about this part, don't let that stop you from continuing to the rest of the question.

(ii) Evaluate $\mathbf{p}_1.\mathbf{p}_1$, $(\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3).\mathbf{p}_1$, $\mathbf{p}_2.\mathbf{p}_1 + \mathbf{p}_3.\mathbf{p}_1$ and $\mathbf{p}_2.\mathbf{p}_1$.

(iii) If the vector $\mathbf{x}$ has length k, show that $(\mathbf{x} - \mathbf{p}_1).(\mathbf{x} - \mathbf{p}_1) = k^2 - 2\mathbf{x}.\mathbf{p}_1 + 1$ and find

$$\sum_{i=1}^{3} (\mathbf{x} - \mathbf{p}_i).(\mathbf{x} - \mathbf{p}_i).$$

(iv) Given that P_1 has coordinates $(0, 1)$, use the values of $\mathbf{p}_2.\mathbf{p}_1$ and $\mathbf{p}_2.\mathbf{p}_2$ to find the coordinates of P_2 and P_3.

4 The STEP III question

The four vertices P_i $(i = 1, 2, 3, 4)$ of a regular tetrahedron lie on the surface of a sphere with centre at O and of radius 1. The position vector of P_i with respect to O is $\mathbf{p}_i$ $(i = 1, 2, 3, 4)$. Use the fact that $\mathbf{p}_1 + \mathbf{p}_2 + \mathbf{p}_3 + \mathbf{p}_4 = \mathbf{0}$ to show that $\mathbf{p}_i \,.\, \mathbf{p}_j = -\frac{1}{3}$ for $i \neq j$.

Let X be any point on the surface of the sphere, and let XP_i denote the length of the line joining X and P_i $(i = 1, 2, 3, 4)$.

(i) By writing $(XP_i)^2$ as $(\mathbf{p}_i - \mathbf{x}) \,.\, (\mathbf{p}_i - \mathbf{x})$, where $\mathbf{x}$ is the position vector of X with respect to O, show that

$$\sum_{i=1}^{4} (XP_i)^2 = 8\,.$$

(ii) Given that P_1 has coordinates $(0, 0, 1)$ and that the coordinates of P_2 are of the form $(a, 0, b)$, where $a > 0$, show that $a = 2\sqrt{2}/3$ and $b = -1/3$, and find the coordinates of P_3 and P_4.

(iii) Show that

$$\sum_{i=1}^{4} (XP_i)^4 = 4 \sum_{i=1}^{4} (1 - \mathbf{x} \,.\, \mathbf{p}_i)^2\,.$$

By letting the coordinates of X be (x, y, z), show further that $\sum_{i=1}^{4} (XP_i)^4$ is independent of the position of X.

Discussion

The very last part is a bit of slog; sorry about that!

It is probably easiest to expand and simplify $\displaystyle\sum_{i=1}^{4} (1 - \mathbf{x} \,.\, \mathbf{p}_i)^2$ before putting in the coordinates of $\mathbf{x}$ and $\mathbf{p}_i$.

You should find that the worst thing you have to calculate is $\left(-\frac{\sqrt{2}}{3}x \pm \frac{\sqrt{2}}{\sqrt{3}}y - \frac{1}{3}z\right)^2$.

STEP Mechanics question

5 Preparation

Two equal thin rods AB and BC, each of length $2a$, are joined at B so that $\angle ABC = 90°$. Let G be the centre of gravity of the combined rods.

(i) Find the coordinates of G with respect to an origin at B, where A has coordinates $(0, 2a)$ and C has coordinates $(2a, 0)$.

(ii) Draw a diagram showing the rods.

Let X be the point on AB such that GX is perpendicular to AB. Find the values of $\tan BGX$ and $\tan AGX$.

(iii) On your diagram, draw the line through B that would be vertical if the rods were hung up by the point B.

On the same diagram, draw the line through A that would be vertical if the rods were hung up by the point A

Let α be the angle between these two lines. Show that $\tan\alpha = -2$.

6 The Mechanics question

A piece of uniform wire is bent into three sides of a square $ABCD$ so that the side AD is missing. Show that if it is first hung up by the point A and then by the point B then the angle between the two directions of BC is $\tan^{-1} 18$.

Discussion

Note how quickly a mechanics question can become a geometry question!

STEP Probability/Statistics question

7 **Preparation**

(i) Sketch the graph $y = \dfrac{3x}{5-2x}$.

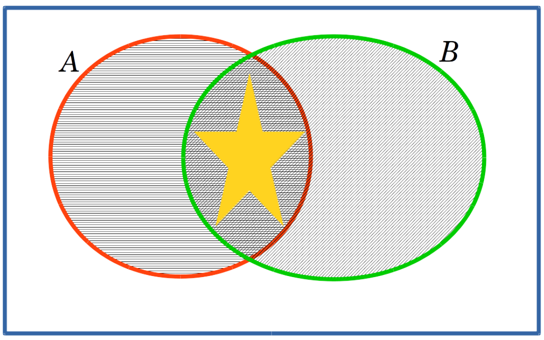
Find the maximum and minimum values of y if x lies in the interval $-1 \leqslant x \leqslant 2$.

To find the horizontal asymptote it may be helpful to divide both the numerator and denominator by x.

(ii) Sketch the graph $y = \dfrac{1-x}{5-2x}$.

Find the maximum and minimum values of y if x lies in the interval $-1 \leqslant x \leqslant 2$.

(iii) The diagram below is called a *Venn diagram*. In a Venn diagram, probabilities are represented by areas, and the blue rectangle (representing the whole sample space) has area 1.



The area in the red circle represents P(A) and the area in the green circle. represents P(B). The probability of A **and** B both happening, which is written as $\mathrm{P}(\mathrm{A} \cap \mathrm{B})$, is represented by the area which lies in both the red and the green circles (the bit with a yellow star in it).

The probability that either A or B, or both, occur is written as $\mathrm{P}(\mathrm{A} \cup \mathrm{B})$ and is represented by the whole area covered by the red and green circles. If A and B were *mutually exclusive* then the two circles do not overlap and $\mathrm{P}(\mathrm{A} \cup \mathrm{B}) = \mathrm{P}(\mathrm{A}) + \mathrm{P}(\mathrm{B})$. If A and B are not mutually exclusive, then we have to ensure that we do not count the intersection twice, so:

$$\mathrm{P}(\mathrm{A} \cup \mathrm{B}) = \mathrm{P}(\mathrm{A}) + \mathrm{P}(\mathrm{B}) - \mathrm{P}(\mathrm{A} \cap \mathrm{B}) \,.$$

The same diagram can be used to find conditional probabilities. In order to find the conditional probability that A happens given that B happens, we look at the fraction of the area of the B circle that is also covered by the A circle. This gives the conditional probability of A happening given that B happens as:

$$P(A|B) = \frac{P(A \cap B)}{P(B)}\,.$$

In order to calculate conditional probabilities when events A and B are not independent, we use:

$$P(A|B) = \frac{P(A \cap B)}{P(B)} \quad \text{and} \quad P(B|A) = \frac{P(A \cap B)}{P(A)}$$

to give:

$$P(A|B)P(B) = P(B|A)P(A)$$

and hence:

$$P(A|B) = \frac{P(B|A)P(A)}{P(B)}$$

This last result is known as **Bayes' Theorem**.

(a) Given that $P(A) = 0.3$, $P(B) = 0.5$, that the events A and B are independent, find $P(A \cup B)$. Find also $P(A|B)$ and $P(B|A)$.
If the events are *independent* then $P(A \cap B) = P(A) \times P(B)$.

(b) Events A and B (not necessarily independent) satisfy:

$$P(A) = 0.6, \quad P(B) = 0.5, \quad P(B|A) = 0.4\,.$$

Find: $P(A|B)$ and $P(A \cup B)$.

(c) 1% of Martians at age forty who participate in routine screening for tentacle rot actually have tentacle rot. It is known that 80% of Martians with tentacle rot will get a positive result in the screening. 10% of Martians without tentacle rot will also get a positive result. A Martian of age forty had a positive result in a routine screening. What is the probability that she actually has tentacle rot?
You can do this is various ways.

- You could just consider a population of 1000 Martians.
- You could draw a Venn diagram.
- You could use Bayes' theorem, in which case you might find the result $P(B) = P(B|A)P(A) + P(B|A')P(A')$ useful. (Here A' is the *complement* of A, i.e. the event "A does not happen".

We suggest that you try all three ways and see which you like best.

8 The Probability question

It is known that there are three manufacturers A, B, C, who can produce micro chip MB666. The probability that a randomly selected MB666 is produced by A is $2p$, and the corresponding probabilities for B and C are p and $1-3p$, respectively, where $0 \leqslant p \leqslant \frac{1}{3}$. It is also known that 70% of MB666 micro chips from A are sound and that the corresponding percentages for B and C are 80% and 90%, respectively.

Find in terms of p, the conditional probability, P(A|S), that if a randomly selected MB666 chip is found to be sound then it came from A, and also the conditional probability, P(C|S), that if it is sound then it came from C.

A quality inspector took a random sample of one MB666 micro chip and found it to be sound. She then traced its place of manufacture to be A, and so estimated p by calculating the value of p that corresponds to the greatest value of P(A|S). A second quality inspector also a took random sample of one MB666 chip and found it to be sound. Later he traced its place of manufacture to be C and so estimated p by applying the procedure of his colleague to P(C|S).

Determine the values of the two estimates and comment briefly on the results obtained.

Discussion

A lot of reading. But you shouldn't be put off by questions that look long on the page: often (and especially for mechanics and probability), they are long because there are quite a few things to say that take up space, but hardly need saying (stuff about no wind resistance, randomness, etc); and sometimes it is because some new concept is being explained — once you have grasped the concept, the question becomes fairly straightforward.