STEP CORRESPONDENCE PROJECT

Assignment 32

STEP I question

1 Preparation

(i) Solve the equation $2\sin^2 x + 3\cos x = 0$ where $0 \leqslant x < 2\pi$.

(ii) Write $\sin^2 x$ in terms of $\cos 2x$. Hence sketch $\sin^2 x$ for $0 \leqslant x \leqslant 2\pi$.

(iii) If $\mathrm{f}(x) = \mathrm{e}^{\cos 2x}$, find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$.

(iv) If $\mathrm{f}(x) = \cos(\tan x)$, find $\mathrm{f}'(x)$ and $\mathrm{f}''(x)$.

(v) Show that $\cot\alpha = \tan(\frac{\pi}{2} - \alpha)$.

(vi) Solve the equation $\sec^2 x = 2\tan x$ for $0 \leqslant x < 2\pi$.

2 The STEP I question

A function $\mathrm{f}(x)$ is said to be *convex* in the interval $a < x < b$ if $\mathrm{f}''(x) \geqslant 0$ for all x in this interval.

(i) Sketch on the same axes the graphs of $y = \frac{2}{3}\cos^2 x$ and $y = \sin x$ in the interval $0 \leqslant x \leqslant 2\pi$.

The function $\mathrm{f}(x)$ is defined for $0 < x < 2\pi$ by

$$\mathrm{f}(x) = \mathrm{e}^{\frac{2}{3}\sin x}.$$

Determine the intervals in which $\mathrm{f}(x)$ is convex.

(ii) The function $\mathrm{g}(x)$ is defined for $0 < x < \frac{1}{2}\pi$ by

$$\mathrm{g}(x) = \mathrm{e}^{-k\tan x}.$$

If $k = \sin 2\alpha$ and $0 < \alpha < \frac{1}{4}\pi$, show that $\mathrm{g}(x)$ is convex in the interval $0 < x < \alpha$, and give one other interval in which $\mathrm{g}(x)$ is convex.

Discussion

In the second part, it will probably be easier to leave k as k (i.e. not use $k = \sin 2\alpha$) until you have finished differentiating. Although the second part does not ask you to draw a sketch, you may find a sketch is useful in the later stages of the question.

STEP III question

3 Preparation

Some remarks on hyperbolic functions

- You shouldn't worry if you haven't covered hyperbolic functions. There is really not much to them; and it is certainly worth learning about them, otherwise you exclude yourself from questions that you might have found doable.
- Hyperbolic functions are defined in terms of exponentials (look it up if you don't know these definitions), in the same way as trigonometric functions could be defined in terms of complex exponentials ($e^{i\theta}$).
- All the formulae for trigonometric functions work similarly for hyperbolic functions provided you remember to change the sign in front of any product of two sinh functions. For example $\cosh^2\theta - \sinh^2\theta = 1$. This is not magic: it follows immediately from the identities $\sinh x = -i\sin(ix)$ and $\cosh x = \cos(ix)$ which themselves follow directly from the definitions (try it if you haven't already done so).
- Although trigonometric and hyperbolic functions are algebraically related as above, they are very distinct geometrically. For example, $\cosh x \geqslant 1$ if x is real; and the equation $\sinh x = 0$ has only one real solution.
- You can write (e.g.) $\cosh^{-1}$ when you get fed up with writing arcosh.

(i) Sketch the graph $y = \tanh x$.

You can differentiate $\dfrac{\sinh x}{\cosh x}$ to find the gradient. You can also express $\tanh x$ in terms of e^x which will help when explaining the behaviour as $x \to \pm\infty$.

(ii) Show that, for $x \geqslant 0$, $\sqrt{\dfrac{\cosh x - 1}{\cosh x + 1}} = \tanh(x/2)$. What is the corresponding result if $x \leqslant 0$?

(iii) By considering $\cosh y = x$, find $\dfrac{d}{dx}\,\text{arcosh}\, x$ in terms of x.

(iv) By using the substitution $\sinh y = x$, find $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,dx$ and check your answer by direct integration.

(v) Give a sketch to show that if $f(t) > g(t)$ for $t > 0$ and $f(0) = g(0)$ then

$$\int_0^x f(t)\,dt > \int_0^x g(t)\,dt$$

and use this result to show that $\frac{1}{2}x^2 > 1 - \cos x$ for $x > 0$.

4 The STEP III question

(i) Show, with the aid of a sketch, that $y > \tanh(y/2)$ for $y > 0$ and deduce that

$$\operatorname{arcosh} x > \frac{x-1}{\sqrt{x^2-1}} \quad \text{for} \quad x > 1. \qquad (*)$$

(ii) By integrating $(*)$, show that $\operatorname{arcosh} x > 2\dfrac{x-1}{\sqrt{x^2-1}}$ for $x > 1$.

(iii) Show that $\operatorname{arcosh} x > 3\dfrac{\sqrt{x^2-1}}{x+2}$ for $x > 1$.

[**Note:** $\operatorname{arcosh} x$ is another notation for $\cosh^{-1} x$.]

Discussion

This is a very nice question: you obtain the first inequality to obtain the second inequality, and the second to obtain the third, and obviously you could keep going, though the denominators get nasty. You are effectively pulling yourself up by your bootstraps.

And just when you feel you can't face another integration, you find that you have already done everything required for part (iii).

But it is not a perfect question, because in fact the inequality of part (ii) could have been obtained from a sketch similar to that of part (i), showing $y/2$ and $\tanh(y/2)$.

STEP Mechanics question

5 **Preparation**

(i) If $\tan\theta = \frac{3}{4}$ and θ is acute, find $\cos\theta$ and $\sin\theta$.

(ii) Here are some thoughts about solving this sort of mechanics problem.

- The first thing to do is to draw a BIG diagram. Draw the wedge with θ much less than 45° so that you can easily see which angles in the diagram are equal to θ and which are equal to $90 - \theta$.
- Put all the forces in your diagram. You may want to draw separate diagrams for the particle and the wedge, so you are not confused about which force acts on what.
- Remember Newton's third law. For example, if the particle experiences a frictional force, then the wedge experiences an equal and opposite frictional force.
- Remember that frictional forces oppose the motion. If you are not sure what direction a frictional force acts in, try to work out which direction the particle would move in if there were no friction.
- For static friction, the equation $F = \mu R$ holds only in limiting equilibrium, when the system is just about to move; but for kinematic friction, when the system is already in motion, $F = \mu R$ is assumed to hold.
- Write down the equations of motion (Newton's second law) for the various objects in the system. But before doing so, have a good look at the result you are aiming for, because it might help you to decide which directions to resolve the forces. For example, there is no g in the first displayed equation in the Mechanics question below: if you weren't sure whether to resolve forces horizontally and vertically, or parallel and perpendicular to the wedge, the absence of g in the equation is a pretty big clue.

6 The Mechanics question

A wedge of mass km has the shape (in cross-section) of a right-angled triangle. It stands on a smooth horizontal surface with one face vertical. The inclined face makes an angle θ with the horizontal surface. A particle P, of mass m, is placed on the inclined face and released from rest. The horizontal face of the wedge is smooth, but the inclined face is rough and the coefficient of friction between P and this face is μ.

(i) When P is released, it slides down the inclined plane at an acceleration a relative to the wedge. Show that the acceleration of the wedge is

$$\frac{a\cos\theta}{k+1}\,.$$

To a stationary observer, P appears to descend along a straight line inclined at an angle 45° to the horizontal. Show that

$$\tan\theta = \frac{k}{k+1}\,.$$

In the case $k = 3$, find an expression for a in terms of g and μ.

(ii) What happens when P is released if $\tan\theta \leqslant \mu$?

Discussion

If you obtained the first displayed equation by resolving forces horizontally for the particle and the wedge separately, you were probably pleased that the reactions and the frictional forces cancelled out. There is (of course) a reason for this. Even though there is friction in the system, so energy (KE+PE) is not conserved, the total momentum of the system is zero, because there are no external horizontal forces acting. Writing down the total horizontal momentum gives the displayed equation. The vertical momentum is not conserved because gravity and the reaction of the horizontal surface on the wedge are external forces.

STEP Probability/Statistics question

7 Preparation

(i) I have a row of 5 current buns, one of which has a cherry on top. If I close my eyes and pick two at random, what is the probability that I pick the bun with a cherry on top?

(ii) Now I have n current buns, of which one has a cherry on top. If I pick r buns, what is the probability that I pick the one with a cherry on top?

You are asked to do the same thing below in part **(iv)**, but here you should use a simple argument.

(iii) The number of ways of choosing r objects from a total of n different objects is $\dfrac{n!}{r!(n-r)!}$, which is written as ${}^nC_r$ or $\binom{n}{r}$. These are also binomial coefficients i.e. the coefficient of x^r in the expansion of $(1+x)^n$.

A team of 5 has to be chosen from a class of 10 swimmers, of whom exactly 2 are male. Find the probability that a team of 5 chosen at random (the probability of each choice being equally likely) will include the 2 males.

You will need to find the total number of ways of choosing 5 from the class of 10, and also (to work out the number of these in which there are two males) the number of ways of picking 3 from the remaining 8 once you have picked the 2 males.

(iv) Use the ${}^nC_r$ notation to find the probability that, if I pick r current buns from a total of n, I get the one with a cherry on top.

You need to consider the total number of ways of picking r buns out of the n as well as the number of ways that include the 1 with a cherry on top.

(v) I have n chocolates of which 2 are filled with chilli and the rest are filled with caramel. If I pick r at random, show that the probability that I manage to avoid the chilli ones is $\dfrac{(n-r)(n-r-1)}{n(n-1)}$.

(vi) Show that ${}^nC_r = {}^{n-1}C_r + {}^{n-1}C_{r-1}$.

This can be done either by manipulating factorials or by a clever "choosing" argument. We suggest that you try both ways. Note that this is the relationship that we see in Pascal's triangle: an entry in a given row is formed by adding together the two closest entries in the row above.

(vii) The sequence a_0, a_1, $\ldots$, a_n has the property that $a_i/a_{i-1} > 1$ for $i \leqslant k$ and $a_i/a_{i-1} < 1$ for $i > k$. What is the largest term of the sequence?

8 The Probability question

Bag P and bag Q each contain n counters, where $n \geqslant 2$. The counters are identical in shape and size, but coloured either black or white. First, k counters ($0 \leqslant k \leqslant n$) are drawn at random from bag P and placed in bag Q. Then, k counters are drawn at random from bag Q and placed in bag P.

(i) If initially $n-1$ counters in bag P are white and one is black, and all n counters in bag Q are white, find the probability in terms of n and k that the black counter ends up in bag P.

Find the value or values of k for which this probability is maximised.

(ii) If initially $n-1$ counters in bag P are white and one is black, and $n-1$ counters in bag Q are white and one is black, find the probability in terms of n and k that the black counters end up in the same bag.

Find the value or values of k for which this probability is maximised.

Discussion