STEP CORRESPONDENCE PROJECT

Assignment 33

STEP I question

1 **Preparation**

Some of this preparation may not be strictly relevant to the way you decide to do the STEP question.

(i) How many different ways can you find to integrate $\displaystyle\int \frac{x}{1-x^2}\,\mathrm{d}x$?

(ii) Show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{y^2}{x} \qquad (x > 0, \quad x^2 \neq \mathrm{e})$$

which satisfies $y = 2$ when $x = 1$ is $y = \dfrac{2}{1 - 2\ln x}$. Sketch this graph.

(iii) Find y in terms of x, given that

$$\frac{y^2}{3x^3} = -\frac{1}{x^2} + 1,$$

and determine the values of x for which y is real.

2 **The STEP I question**

(i) Use the substitution $y = ux$, where u is a function of x, to show that the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$ is

$$y = x\sqrt{4 + 2\ln x} \qquad (x > \mathrm{e}^{-2}).$$

(ii) Use a substitution to find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

(iii) Find the solution of the differential equation

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{x^2}{y} + \frac{2y}{x} \qquad (x > 0,\ y > 0)$$

that satisfies $y = 2$ when $x = 1$.

Discussion

You should explain why the positive square root is taken, and also you should state the domain for which the solutions hold, such as given in part **(i)** where the domain is given as $x > \mathrm{e}^{-2}$.

You can look for a different substitution for part **(ii)** if you like, but you don't have to.

You will need a substitution for part **(iii)** as well, and you will find that the substitution $y = ux$ won't give a differential equation in which the variables are separable. There are several ways to proceed, including finding a suitable substitution. A bit of trial and error might be necessary, but do remember that these questions are supposed to take 30 to 45 minutes so you will have time to experiment. It will help to have a good look at the substitution in part **(ii)** to see how it works.

STEP III question

3 Preparation

(i) If $\mathrm{f}(0) = 0$ and $\mathrm{f}'(x) \geqslant 0$ for $x \geqslant 0$ what can you say about $\mathrm{f}(x)$? (Be careful to make an accurate statement!)

(ii) Differentiate with respect to x:

(a) $\mathrm{e}^{\sin x}$;
(b) $\cosh(x^2)$
(c) $\sinh(\mathrm{f}(x))$, where f is a given function with derivative f';
(d) $x\mathrm{f}'(x)$;
(e) $(\mathrm{f}'(x))^2$.

(iii) Integrate $Ax\sin x + B\cos x$ with respect to x;

(iv) Integrate $\mathrm{f}(x)\mathrm{e}^{\mathrm{f}(x)}\mathrm{f}'(x)$ with respect to x.

(v) By writing the hyperbolic functions in terms of exponentials, show that $\cosh x \geqslant 1$.

4 The STEP III question

(i) Let $y(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2y}{\mathrm{d}x^2}+y^3=0$ with $y=1$ and $\dfrac{\mathrm{d}y}{\mathrm{d}x}=0$ at $x=0$, and let

$$\mathrm{E}(x)=\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2+\tfrac{1}{2}y^4\,.$$

Show by differentiation that E is constant and deduce that $|y(x)|\leqslant 1$ for all x.

(ii) Let $v(x)$ be a solution of the differential equation $\dfrac{\mathrm{d}^2v}{\mathrm{d}x^2}+x\dfrac{\mathrm{d}v}{\mathrm{d}x}+\sinh v=0$ with $v=\ln 3$ and $\dfrac{\mathrm{d}v}{\mathrm{d}x}=0$ at $x=0$, and let

$$\mathrm{E}(x)=\left(\frac{\mathrm{d}v}{\mathrm{d}x}\right)^2+2\cosh v\,.$$

Show that $\dfrac{\mathrm{dE}}{\mathrm{d}x}\leqslant 0$ for $x\geqslant 0$ and deduce that $\cosh v(x)\leqslant \frac{5}{3}$ for $x\geqslant 0$.

(iii) Let $w(x)$ be a solution of the differential equation

$$\frac{\mathrm{d}^2w}{\mathrm{d}x^2}+(5\cosh x-4\sinh x-3)\frac{\mathrm{d}w}{\mathrm{d}x}+(w\cosh w+2\sinh w)=0$$

with $\dfrac{\mathrm{d}w}{\mathrm{d}x}=\dfrac{1}{\sqrt{2}}$ and $w=0$ at $x=0$. Show that $\cosh w(x)\leqslant\frac{5}{4}$ for $x\geqslant 0$.

Discussion

Don't attempt to solve any of the differential equations in this question! They are all non-linear (i.e. they depend on y^2 or worse), and in general such equations can only be solved numerically by computer.

Even the equation in part (i), which doesn't look too bad, can only be solved in terms of what are called *elliptic* functions.

This is not an easy question, not because any one step is hard, but because there are lots of steps each of which needs an idea which may well be unfamiliar.

STEP Mechanics question

5 Preparation

This mechanics question only requires fairly simple manipulations (differentiation, scalar product) of vectors, so there is not really any preparation to be done; just get on with it!

Of course, to find the derivative of a vector, you just differentiate its components.

6 The Mechanics question

A particle P moves so that, at time t, its displacement $\mathbf{r}$ from a fixed origin is given by

$$\mathbf{r} = \left(\mathrm{e}^t \cos t\right)\mathbf{i} + \left(\mathrm{e}^t \sin t\right)\mathbf{j}\,.$$

Show that the velocity of the particle always makes an angle of $\frac{\pi}{4}$ with the particle's displacement, and that the acceleration of the particle is always perpendicular to its displacement. Sketch the path of the particle for $0 \leqslant t \leqslant \pi$.

A second particle Q moves on the same path, passing through each point on the path a fixed time T after P does. Show that the distance between P and Q is proportional to e^t.

Discussion

The statement made in the 'preparation' section: 'Of course, to find the derivative of a vector, you just differentiate its components.' is not actually an 'of course' statement. You need to think about, for example.

$$\lim_{h\to 0} \frac{\mathbf{r}(t+h) - \mathbf{r}(t)}{h}$$

which is the definition of the derivative. Then you write it out in terms of components to get the result stated above. You are not expected to do that here.

The result may be obvious, but does need justification; in fact, it would not be correct if the vector were written in polar coordinates and axes.

STEP Probability/Statistics question

7 Preparation

(i) Suppose that we want to make a choice from 3 options, P, Q and R, but only have a fair coin to do this.

One option is to toss the coin twice and then:

- if we get HH choose P,
- if we get HT choose Q,
- if we get TH choose R,
- and if we get TT then repeat, i.e. toss the coin twice again.

(a) What is the probability that we choose P on the first two tosses of the coin?

(b) What is the probability that we don't make a choice on the first two tosses, but choose P on the second two tosses?

(c) Write down an infinite sum for the probability that we eventually choose P and hence find the probability that we choose P. Comment on your answer.

(ii) This time we keep tossing the coin until we get one of the following sequences:

- if HH appears first, choose P,
- if HT appears first, choose Q,
- and if TH appears first, choose R.

So if we have the sequence "TTTTTH" we would choose R as out of the three sequences, TH appears first.

By considering what happens if the first toss is a Head or a Tail, find the probabilities that you choose P, Q and R.

8 The Probability question

Four players A, B, C and D play a coin-tossing game with a fair coin. Each player chooses a sequence of heads and tails, as follows:

Player A: HHT; Player B: THH; Player C: TTH; Player D: HTT.

The coin is then tossed until one of these sequences occurs, in which case the corresponding player is the winner.

(i) Show that, if only A and B play, then A has a probability of $\frac{1}{4}$ of winning.

(ii) If all four players play together, find the probabilities of each one winning.

(iii) Only B and C play. What is the probability of C winning if the first two tosses are TT?

Let the probabilities of C winning if the first two tosses are HT, TH and HH be p, q and r, respectively. Show that $p = \frac{1}{2} + \frac{1}{2}q$.

Find the probability that C wins.

Discussion

It is hard to write a preparation for this question which does not give too much away. For the first part think carefully about who must win after the first two tosses in the 4 different cases.

For the last part, you are asked to derive a given equation; then you need to find two more equations.