STEP CORRESPONDENCE PROJECT

Assignment 5

Warm-up

1 **(i)** Evaluate $\sum_{n=0}^{6} \sin(\frac{n\pi}{6})$.

Note: $\frac{n\pi}{6}$ is in radians; 'evaluate' means give the (exact) value — so leave it in the form of surds.

(ii) Let $S_n = \sum_{i=0}^{n-1} r^i$ (which is n terms altogether). Simplify $rS_n - S_n$ and hence find a formula, in the case $r \neq 1$, for S_n. What is the corresponding result when $r = 1$?

Deduce a formula for $\sum_{i=0}^{n-1} ar^i$.

Hint: If you are not very used to working with the $\sum$ notation, you may prefer to write out the sums with dots (for example $1 + r + r^2 + \cdots + r^{n-1}$). It is conventional to use exactly three dots.

(iii) Evaluate $\sum_{i=0}^{9} 3\left(\frac{1}{2}\right)^i$.

Preparation

2 **(i)** Sketch the graph of $y = 2x + 1$. Use your graph and the formula for the area of a trapezium (or some similar formulae if you prefer) to evaluate

$$\int_2^5 (2x + 1)\,dx\,.$$

Check your answer by integral calculus (i.e. 'do' the integral!).

(ii) The square bracket ('floor')[1] notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi] = 3$, $[\sqrt{24}\,] = 4$ and $[5] = 5$.

What is $[10.2]$, $[\sqrt{70}]$, $[6]$ and $[10\pi]$?

If $3 \leqslant x < 4$, what is $[x]$?

(iii) Sketch the graph of $y = [x]$ for $0 \leqslant x < 4$. Use your sketch to evaluate

$$\int_0^4 [x]\, dx\,.$$

(iv) Sketch the graph of $y = x[x]$ for $0 \leqslant x < 3$. Use your sketch to evaluate

$$\int_0^3 x[x]\, dx\,.$$

The STEP question

3 The square bracket notation $[x]$ means the greatest integer less than or equal to x. For example, $[\pi] = 3$, $[\sqrt{24}\,] = 4$ and $[5] = 5$.

(i) Sketch the graph of $y = \sqrt{[x]}$ and show that

$$\int_0^a \sqrt{[x]}\, \mathrm{d}x = \sum_{r=0}^{a-1} \sqrt{r}$$

when a is a positive integer.

(ii) Show that $\displaystyle\int_0^a 2^{[x]}\, \mathrm{d}x = 2^a - 1$ when a is a positive integer.

(iii) Determine an expression for $\displaystyle\int_0^a 2^{[x]}\, \mathrm{d}x$ when a is positive but not an integer.

Discussion

See next page.

[1]The square bracket $[a]$ and the floor bracket $\lfloor a \rfloor$ are equivalent notations for 'integer part'. Carl Friedrich Gauss introduced the square bracket notation in his third proof of quadratic reciprocity (1808). This remained the standard notation in mathematics until Kenneth E. Iverson introduced the names "floor" and "ceiling" and the corresponding notations in his 1962 book A Programming Language.

Discussion

This is very typical of STEP: the question involves a function (or process or notation) that is — or may be — unfamiliar to you but which is defined carefully in the question; then you are asked to use the function for progressively more difficult tasks. You should not be put off by this. It may take a little time to understand the definition, but then you should be able to apply it with confidence.

It is also typical in that there are quite a few disparate concepts in a single question: integer part, integration (as area), summation notation and geometric progression.

Yet again, clear diagrams are invaluable. The last part is rather clever: once you have got into the question, you will find that you need to use the floor function again.

Warm down

4 Arthur, Brenda, and Chandrima, together with an orangutan, are shipwrecked on a desert island. They spend the first day gathering a total of N bananas which they heap up together into one pile. Then they go to sleep for the night.

In the middle of the night, Arthur wakes up. He decides to take his share immediately in case there is an argument about it in the morning. He divides the bananas into three equal piles but finds that there is one banana left over. This he gives to the orangutan. He then hides his pile, heaps the rest all back together and goes back to sleep.

Shortly afterwards, Brenda wakes up and does the same thing, dividing the pile that she finds into three equal piles and hiding her pile. She also has one left over, which she gives to the orangutan before going back to sleep.

Shortly after this, Chandrima wakes up and does the same, taking one third of the bananas in the pile she finds when she wakes up after giving one to the orangutan.

In the morning they divide what bananas are left, which comes out in three equal shares of m bananas, with one left over for the orangutan. Of course each one knows that there are bananas missing; but each one is as guilty as the others, so no one says anything.

Show that

$$8N = 81m + 65 \qquad (*)$$

Show that if N satisfies $(*)$ for some integer m, then $N + 81$ is a solution (for a different value of m).

Explain how the division into piles works in the case the $N = -2$ (two negative bananas) and find a solution for positive bananas.

If you have time, you might like to find a general formula for the integers N and m that can satisfy $(*)$ (you would need to show that there are no others).