STEP CORRESPONDENCE PROJECT

Assignment 7

Warm-up

1 **(i)** Sketch (on different axes) the graphs of $y = x + \dfrac{1}{x}$ and $y = x - \dfrac{1}{x}$ (for $x \neq 0$), paying particular attention to:

(a) the location of turning points (local maxima/minima), if there are any;

(b) the behaviour as $x \to \infty$ and $x \to -\infty$;

(c) the behaviour when x is close to zero;

(d) the location of intercepts (places where the graph crosses the x or y axes), if there are any.

(ii) For what values of x is

(a) $x + \dfrac{1}{x} > 2$?

(b) $x - \dfrac{1}{x} \geqslant \frac{3}{2}$?

Preparation

2 One of the most difficult things for first year university students is trying to find a proof for something that seems very obvious.

The key is always to go back to definitions. In university mathematics, we always start with definitions, and any statement or theorem has to be proved using those definitions.[1]

In this question, *forget that you know how to solve a quadratic equation.* It may seem strange that we are asking you to forget, for the moment, something that everyone else is trying to make you remember; but the idea is to come up with methods that could be extended to more complicated equations.

(i) Show that if α and β (where $\alpha \neq \beta$) both satisfy the quadratic equation

$$x^2 + bx + c = 0$$

(so $\alpha^2 + \alpha b + c = 0$, and similarly for β) then $b = -(\alpha + \beta)$. Find an expression for c in terms of α and β.

Hence show that $(x - \alpha)(x - \beta) \equiv x^2 + bx + c$.

Note: The symbol "$\equiv$" means that the relationship holds for all values of x (it is an *identity*). It is said 'is equivalent to'. For example we can write $3x + 3 \equiv 3(x + 1)$ as for any value of x two sides have the same value; (the two sides are just two ways of writing the same thing). However if we write $3x + 3 = 6$, using an equals sign rather than an equivalence sign, we mean that the statement may only hold for some particular value or values of x (in this case only when $x = 1$).

(ii) Now we do the reverse of part (i).

Given that $x^2 + bx + c \equiv (x - \alpha)(x - \beta)$, substitute $x = 0$ into both sides and hence show that $\alpha\beta = c$. By substituting $x = 1$ into both sides, find an expression for $\alpha + \beta$.

(iii) Write down two positive integers α and β such that $\alpha\beta = 10$ and $\alpha + \beta = 7$. Hence write down the roots of the equation $x^2 - 7x + 10 = 0$.

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[1] Sometimes the results are 'obvious' and sometimes they are startling.

For example, the theorem (called Rolle's theorem) that if $f(a) = 0$ and $f(b) = 0$, then $f(x)$ has a turning point between a and b seems extremely obvious: if $f(x)$ didn't have a turning point, it couldn't get back to 0. But it still has to be proved, starting from the definition of a differentiable function.

A startling example is the theorem (Banach-Tarski) which shows, starting from some pretty harmless definitions, you can cut a solid sphere into several pieces then put them back again to form a smaller solid sphere! Of course, the cuts have to be very peculiar.

(iv) It is given that $x^3 + bx^2 + cx + d \equiv (x - \alpha)(x - \beta)(x - \gamma)$. By substituting 3 different values of x show that:

$$\alpha\beta\gamma = -d$$
$$(1 - \alpha)(1 - \beta)(1 - \gamma) = 1 + b + c + d$$
$$(1 + \alpha)(1 + \beta)(1 + \gamma) = 1 - b + c - d$$

(v) Given that

$$x^3 - 4x^2 - 4x + 16 \equiv (x - \alpha)(x - \beta)(x - \gamma)\,,$$

where α, β and γ are integers, show using one of the equations from part (iv) that:

$$(1 + \alpha)(1 + \beta)(1 + \gamma) = -15$$

and hence find the four sets of values that $\{1 + \alpha\,,\ 1 + \beta\,, 1 + \gamma\}$ can take.

For example, one set of values is $1 + \alpha = 5$, $1 + \beta = 3$, $1 + \gamma = -1$. Note that the set of values $1 + \alpha = 3$, $1 + \beta = 5$, $1 + \gamma = -1$ does not count as a different set of values.

Hence find the possible sets of values of $\{\alpha, \beta, \gamma\}$ and, by testing them all in the equation $\alpha\beta\gamma = -d$, find α, β and γ. Hence write down the roots of the equation $x^3 - 4x^2 - 4x + 16 = 0$.

The STEP question

3 Let

$$\mathrm{f}(x) = x^n + a_1x^{n-1} + \cdots + a_n\,,$$

where a_1, a_2, $\ldots$, a_n are given numbers. It is given that $\mathrm{f}(x)$ can be written in the form

$$\mathrm{f}(x) = (x + k_1)(x + k_2)\cdots(x + k_n)\,.$$

By considering $\mathrm{f}(0)$, or otherwise, show that $k_1k_2\ldots k_n = a_n$.

Show also that

$$(k_1 + 1)(k_2 + 1)\cdots(k_n + 1) = 1 + a_1 + a_2 + \cdots + a_n$$

and give a corresponding result for $(k_1 - 1)(k_2 - 1)\cdots(k_n - 1)$.

Find the roots of the equation

$$x^4 + 22x^3 + 172x^2 + 552x + 576 = 0\,,$$

given that they are all integers.

Discussion

When you put $x = -1$ into $\mathrm{f}(x)$, you may find it easier to start at the 'other end', by writing

$$\mathrm{f}(x) = a_n + \cdots + a_1 x^{n-1} + x^n \,.$$

In STEP questions, you are not allowed a calculator but might still be asked to work with large-ish numbers. You will find that one of the three equations (obtained from setting $x = 0$, etc) is easier to work with initially (it has fewer factors).

Warm down

4 I have a very old set of balancing scales.

(i) If I can only put the weights in one of the scale pans:

(a) Which weights do I need if I wish to be able to weigh out all the integer number of ounces from 1 ounce to 31 ounces?

(b) Show that if I have only n weights, I cannot weigh more than 2^n different weights (including zero ounces). Hint: each weight is either in the pan or not in the pan.

(ii) If I can put the weights in either of the scale pans:

(a) Which two weights do I need if I wish to be able to weigh 1 ounce, 2 ounces, 3 ounces and 4 ounces?

(b) Show that if I have only n weights, I cannot weigh more than 3^n different weights (including zero ounces).

(c) Which weights do I need if I wish to be able to weigh out all the integer number of ounces from 1 ounce to 40 ounces.