STEP CORRESPONDENCE PROJECT

Assignment 8

Warm-up

1 By considering $(x-y)^2$, prove that $x^2+y^2 \geqslant 2xy$ and hence show that, if a and b are non-negative numbers, then

$$\frac{a+b}{2} \geqslant \sqrt{ab}. \qquad (*)$$

What can you say about a and b if $\frac{1}{2}(a+b)=\sqrt{ab}$?

General remark: if you are trying to show that $A \geqslant B$, it is often easiest to start by considering $A - B$.

The inequality $(*)$ is called the AM-GM inequality: AM stands for *Arithmetic Mean* (the left hand side of $(*)$) and GM stands for *Geometric Mean* (the right hand side of $(*)$). Thus the inequality $(*)$ says that for two non-negative numbers, AM is not less than GM. This is an extremely powerful result, which extends — as we shall see below — to more than two numbers.

(i) Prove that $x^2+y^2+z^2 \geqslant xy+yz+zx$. What can you say about x, y and z when equality holds in this equation?

You might like to start by multiplying both sides by 2.

(ii) Write out the following, giving careful explanations of each step (a typical explanation might be '... using $(*)$ with $a=q$ and $b=r$'):

$$\frac{p+q+r+s}{4} \geqslant \frac{\sqrt{pq}+\sqrt{rs}}{2} \geqslant \sqrt[4]{pqrs}$$

Hence AM $\geqslant$ GM for four numbers. For this and the next part, you should lay out your work carefully, with a new line for each step.

(iii) Write out the following, giving careful explanations of each step (you will need to use the result of part (ii) above):

$$\frac{p+q+r}{3} = \frac{p+q+r}{4} + \frac{p+q+r}{12} \geqslant \sqrt[4]{pqr\left(\frac{p+q+r}{3}\right)}\,.$$

Deduce that

$$\frac{p+q+r}{3} \geqslant \sqrt[3]{pqr}\,.$$

Hence AM $\geqslant$ GM for three numbers. It seems strange to prove the result for the seemingly more complicated four-number case first, then work back to the three-number case; in fact, this is the basis of the very beautiful method ('backward induction') used by Cauchy to prove the general n-number case.

Preparation

2 **(i)** Find the equation of the circle which has the line joining $(1, 5)$ to $(-5, 13)$ as its diameter.

(ii) Find the (real) solutions of the following simultaneous equations:

(a) $x^2 + y^2 = 25$ and $x^2 + (y - 7)^2 = 18$

(b) $x^2 + y^2 = 25$ and $x^2 + (y - 7)^2 = 4$

(c) $x^2 + y^2 = 25$ and $x^2 + (y - 7)^2 = 1$

In each case, illustrate the result by sketching a diagram.

(iii) For the equations in part ii(a) above, show that, if a circle with centre (a, b) passes through the two points of intersection, then $a = 0$. Find the equation of such a circle with centre $(0, 2)$.

(iv) Solve the simultaneous equations

$$x^2 + y^2 = 4$$
$$8x^2 + 4(y - 1)^2 = 36$$

You may like to sketch these equations using Wolfram Alpha or something similar to see what is happening.

The STEP question

3 Show that the equation of any circle passing through the points of intersection of the ellipse $(x + 2)^2 + 2y^2 = 18$ and the ellipse $9(x - 1)^2 + 16y^2 = 25$ can be written in the form

$$x^2 - 2ax + y^2 = 5 - 4a\ .$$

Don't be put off by the word 'ellipse' in the question: you don't have to know anything at all about ellipses to do the question.

Also, don't be put off by the rather large coefficents in the quadratic equation you get when finding the points of intersection of the two ellipses: you can — and should — solve the equations without resorting to calculators or internet quadratic equation solvers.

Warm down

4 I have a drawer full of identical red socks, identical blue socks and identical green socks.

(i) How many socks must I take to ensure that I have a matching pair?

(ii) How many socks must I take to ensure that I have two pairs?

(iii) How many socks must I take to ensure that I have n matching pairs?

There are many ways of tackling this. One approach is to consider picking $2n$ socks, made up of r red socks, b blue socks and g green socks, and decide whether this is enough, in the cases that arise according to whether r, b and g are odd or even and, if it is not enough, determine how many more are required.

However you tackle it, you should explain your answer carefully. Extrapolation from the previous two parts provides guidance and insight, but is not sufficient as a proof.