STEP CORRESPONDENCE PROJECT

Assignment 9

Warm-up

1 **(i)** The isosceles triangle $\triangle ABC$ has $AB = BC$. A line is drawn connecting B to the midpoint M of AC. By considering congrugent triangles show that this line is perpendicular to AC, that it bisects the angle at B, and that $\angle BAC = \angle BCA$.

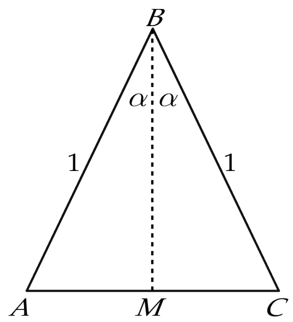
(ii) The triangle ABC has sidelengths $BC = a$, $CA = b$ and $AB = c$. Given that the area of a triangle can be expressed as $\frac{1}{2}$ base $\times$ height, show (by taking the base to be AC) that

$$\text{Area} = \tfrac{1}{2}ba\sin C\,.$$

Show also that Area $= \frac{1}{2}bc\sin A$ and deduce the *sine rule*

$$\frac{\sin A}{a} = \frac{\sin C}{c} = \frac{\sin B}{b}\,.$$

(iii) $\triangle ABC$ is shown below, with $AM = MC$, $AB = BC = 1$ and $\angle ABM = \angle CBM = \alpha$.



Show that $AC = 2\sin\alpha$, and (by using the sine rule) that $\sin(2\alpha) = 2\sin\alpha\cos\alpha$.

Preparation

2 **(i)** Solve the equation $5x^2 + 3x = 0$

(ii) Consider the graph of $y = x^3 - 12x + 1$.

(a) Find the x coordinates of the turning points.

(b) By considering the shape of the graph, state which of the turning points is the maximum and which is the minimum.

(c) Find where the graph intersects the y-axis.

(d) Find the y coordinates of the turning points and sketch the graph.

(e) How many real roots of the equation $x^3 - 12x + 1 = 0$ are there? We are not asking you to find the roots!

(iii) Sketch the graph of each of the following cubics, labelling the coordinates of the turning points and the y intercepts. Do not attempt to find the intercepts with the x-axis; do not use a graphical calculator; do not plot points (other than turning points and the y-intercept).

(a) $y = x^3 + 3x^2 + 1$

(b) $y = 2x^3 + 6x^2 - 3$

(c) $y = 4x^3 + 6x^2 - 3$

(d) $y = x^3 - 6x^2 + 2$

(e) $y = 2x^3 - 3x^2 + 2$

(f) $y = x^3 - 12x^2 - 6$

(iv) Which of the above graphs have three x intercepts?

Consider the locations of the turning points in relation to the x-axis (that is above or below the axis). What is the difference between the graphs with three x intercepts and the graphs with one x intercept?

The STEP question

3 Sketch the curve

$$f(x) = x^3 + Ax^2 + B$$

first in the case $A > 0$ and $B > 0$, and then in the case $A < 0$ and $B > 0$.

Show that the equation

$$x^3 + ax^2 + b = 0,$$

where a and b are real, will have three distinct real roots if

$$27b^2 + 4a^3b < 0,$$

but will have fewer than three if

$$27b^2 + 4a^3b > 0.$$

Discussion

For the first part you should find the coordinates of the turning points in terms of A and B to use on your graphs. In the case $A < 0$ and $B > 0$, there are two sub-cases to consider, according to the sign of the y coordinate of the minimum point.

For the second part, you may find it helpful to factorize $27b^2 + 4a^3b$. No serious calculations are required: the results follow from appropriate sketches.

Warm down

Euclid was a Greek mathematician who lived roughly from the middle of the fourth century BC to the middle of the third century BC (about 100 years after Plato). In his treatise *The Elements*, he builds up what is now called Euclidean geometry in a logical order, by means of a series of propositions each one relying only on previous propositions.

Euclid's Elements were used as a basis for teaching geometry for 23 centuries (in fact until about 1960 when some mathematical educationalists thought they knew better). The logical structure of Euclid's Elements is the model for the teaching of other subjects, notably university level mathematical analysis which lays the foundations of calculus.

Euclid's Proposition 5, which is proved in part (i) of the warm-up above, says that the base angles of an isosceles triangle are equal. Euclid's proof was a bit harder than the one suggested in part (i) above, because he had not yet (i.e. in propositions 1 to 4) shown that two triangles with corresponding sides of the same length are congruent (what we call SSS). However, he had shown (though perhaps not satisfactorily) that the SAS condition is enough for two triangles to be congruent and this is what he used in the proof below.

Because Proposition 5 is the first proposition for which the proof is not straightforward, it is called the Pons Asinorum (bridge of donkeys) — presumably meaning a bridge to the later propositions that separates mathematical donkeys from more fleet-footed mathematicians.

4 The isosceles triangle $\triangle ABC$ has $AB = BC$. The lines BA and BC are extended by equal lengths to D and E respectively. We want to show that the angles BAC and BCA (indicated by rather pretty stars) are equal, using only congruence established with SAS.

First show that angles BCD and BAE are equal (by showing that two triangles are congruent).

Then show that angles DCA and EAC are equal, and deduce the required result.